A
NEW
INTERMEDIATE ALGEBRA

INTENDED FOR THE USE OF INTERMEDIATE CLASSES
IN INDIAN COLLEGES

BY

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PREFACE

It is with great pleasure that we lay before the public a New Intermediate Algebra. About two years and a half ago we brought out an Elementary Matriculation Algebra; and the very kind reception that was accorded to that work by Mathematical Teachers encouraged us in the publication of this work, which is practically a continuation of the previous work. Many chapters from the Elementary Matriculation Algebra, with slight additions and alterations, have been incorporated in it, and other topics have been added which make this book cover the Intermediate syllabuses of all Indian Universities.

The method that we adopted in the previous work has been adhered to here also, *viz.*, that we have tried to lay special stress on the elucidation of the principles of the subject and to illustrate them by a large variety of examples suitably chosen, both worked out and given as exercises for students.

We shall be happy if this book satisfactorily meets the requirements of the Intermediate students of our Colleges, and succeeds in meeting with the approval of their Teachers.

Exigencies of rapid printing have unfortunately resulted in a number of errors and misprints. Those which have caught our eyes we have drawn attention to in a table of *errata* at the end of the book. Any notices of further errors and misprints, and any suggestions for improvement will be thankfully accepted.

JULY 4, 1929
CALCUTTA.

DEVAPRASAD GHOSH
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PREFACE TO THE THIRD EDITION

In this edition the book has been subjected to a thorough revision. It has also been partially recast, for some of the matter which was not very useful to Intermediate students has been omitted, some changes (and it is to be hoped improvements) made in the presentation of arguments in many cases, and many recent University questions incorporated in the body of the book to familiarize the students with the trend of present-day examinations. It is to be hoped that these changes will increase the usefulness of the book to those for whom it is intended.

Any suggestion of improvement and any notices of errors or misprints will be thankfully received.

JULY 14, 1912
CALCUTTA.

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CALCUTTA UNIVERSITY
INTERMEDIATE SYLLABUS IN ALGEBRA

Theory of Quadratic Equations and expressions.

Simultaneous Quadratic Equations, one of which is linear.

Permutations and Combinations.

Variation.

Binomial Theorem for any rational index.

Theory of Indices.

Surds and Complex Quantities.

Logarithms, and their simple applications to Interest and Annuities.

Exponential and Logarithmic Series.

(The following chapters cover the Syllabus : Chaps. I, II, IV, VI, VIII, IX, XI, XII, XIV, XV, XVI, XVII, XVIII.)

CONTENTS

CHAPTER I

PRELIMINARY

1. We propose in this Preliminary Chapter to re-state some of the more important definitions, theorems and results which the student has already learnt in Elementary Algebra, but which it will be convenient to set out here for purposes of reference.

2. Functions.

Any quantity whose value depends upon that of another quantity is said to be a *function* of that quantity. Thus, since the value of x^2 depends upon that of x, x^2 is a function of x.

Hence *any expression involving x is a function of x.*

Similarly, if an expression involves $x, y, z\cdots\cdots$ then it is said to be a function of $x, y, z, \cdots\cdots$; thus, $x^2+y^2+z^2-yz-zx-xy$ is a function of x, y, z.

The quantities upon which the value of the function depends are called *independent variables*, while the function itself is called the *dependent variable.*

The independent variables are sometimes shortly called simply *variables*, and the function is then described as *a function of these variables.*

Quantities other than those considered as variables occurring in an expression or function are described as *constants.* Thus, in the expression x^2+5x+6, x is the variable and 5 and 6 are constants; in the expression ax^2+bx+c, if x is considered the only variable, a, b, c are considered constants.

Usually, letters from the beginning of the alphabet $a, b, c\ldots$ are used to denote constants, and letters towards the end $x, y, z\ldots$ are used to denote variables.

3. Functional Notation.

When an expression is a function of x if it is merely intended to indicate the relation of dependence or functionality, the expression is written $f(x)$. Writing an expression in the form $f(x)$, however, does not give us any idea as to what the nature of the function is, but merely indicates that the expression involves x.

In any particular context, however, $f(x)$ always means one particular kind of function ; thus, in any example, if

$f(x)$	means	$x^2+3x+10$,
$f(0)$	will mean	$0^2+3.0+10$,
$f(1)$	,, ,,	$1^2+3.1+10$,
$f(2)$	,, ,,	$2^2+3.2+10$, and so on.

That is to say, $f(0)$, $f(1)$, $f(2)$,......will involve 0, 1,2······in exactly the same way as $f(x)$ involves x.

Sometimes $F(x)$ and $\phi(x)$ (ϕ being a Greek letter called *phi*) are used to denote functions of x.

Similarly, a function of $x, y, z,\ldots$ is denoted by $f(x, y, z,\ldots)$, or $F(x, y, z,\ldots\ldots)$, or $\phi(x, y, z,\ldots\ldots)$.

4. Remainder Theorem.

If a rational integral function of x be divided by $x-a$, the remainder is obtained by substituting a for x in the function.

Let $f(x)$ be a rational integral function of x, *i.e.*, a function involving only positive integral powers of x.

When $f(x)$ is divided by $x-a$, let Q be the quotient and R the remainder. Then we have

$$f(x)=(x-a)Q+R,$$

where R evidently does not involve x.

Since this relation holds good for all values of x, it will hold good when $x=a$. Then the relation becomes

$$f(a)=(a-a)Q'+R \text{ (where } Q' \text{ is the value of } Q \text{ when } x=a)$$
$$=0\times Q'+R=R.$$

Hence, R, the remainder when $f(x)$ is divided by $x-a$, is $f(a)$, *i.e.*, the value of the given function when a is substituted for x.

Thus, if x^2+px+q is divided by $x-a$, the remainder is a^2+pa+q ; if x^3-2x^2+3x-1 is divided by $x-2$, the remainder is $2^3-2.2^2+3.2-1=5$.

5. Factor Theorem.

If a rational integral function of x becomes zero when a is substituted for x, then $x-a$ is a factor of the function.

It follows at once from the Remainder Theorem, and the student will find it very useful in factorization by trial.

6. Theorems in Divisibility.

The following theorems in divisibility follow at once from the Remainder Theorem :

(*i*) *$x^n - y^n$ is divisible by $x-y$ for all values of n.*

The result of the division is shown below :

$$\frac{x^n - y^n}{x-y} = x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + x^2y^{n-3} + xy^{n-2} + y^{n-1}.$$

(*ii*) *$x^n - y^n$ is divisible by $x+y$, only if n is an even number.*

The result of the division is shown below :

$$\frac{x^n - y^n}{x+y} = x^{n-1} - x^{n-2}y + x^{n-3}y^2 \cdots \cdots - x^2y^{n-3} + xy^{n-2} - y^{n-1}.$$

(*iii*) *$x^n + y^n$ is divisible by $x+y$, only if n is an odd number.*

The result of the division is shown below :

$$\frac{x^n + y^n}{x+y} = x^{n-1} - x^{n-2}y + x^{n-3}y^2 \cdots\cdots + x^2y^{n-3} - xy^{n-2} + y^{n-1}.$$

(*iv*) *$x^n + y^n$ is not divisible by either $x+y$ or $x-y$, if n is an even number, but a remainder is left.*

The result of the division is shown below :

$$\frac{x^n + y^n}{x+y} = x^{n-1} - x^{n-2}y + x^{n-3}y^2 \cdots\cdots + xy^{n-2} - y^{n-1} + \frac{2y^n}{x+y}.$$

$$\frac{x^n + y^n}{x-y} = x^{n-1} + x^{n-2}y + x^{n-3}y^2 \cdots\cdots + xy^{n-2} + y^{n-1} + \frac{2y^n}{x-y}.$$

7. Representation of Even and Odd Numbers.

An *even* number in general is represented by $2n$, where n is an integer ; since for all integral values of n, $2n$ gives some even number or other, but never an odd number.

An *odd* number in general is represented by $2n+1$, where n is an integer ; since for all integral values of n, $2n+1$ gives some odd number or other, but never an even number.

The even numbers in succession can be obtained by putting $n=1, 2, 3, \cdots\cdots$ in $2n$; and the odd numbers in succession can be obtained by putting $n=0, 1, 2, 3, \cdots\cdots$ in $2n+1$.

Sometimes $2n-1$ is used to denote an odd number in general; in that case, the odd numbers in succession can be obtained by putting $n=1, 2, 3, \cdots\cdots$ in $2n-1$.

The student will also find it useful to remember that -1 raised to an even power is equal to $+1$, and raised to an odd power is equal to -1. In other words,

$$(-1)^{2n}=+1, \ (-1)^{2n+1}=(-1)^{2n-1}=-1.$$

8. Imaginary Quantities.

Sometimes in the course of algebraical processes we get square roots of negative quantities; *e. g.* $\sqrt{-1}$, $\sqrt{-2}$, $\sqrt{-3}$, etc. Since the squares of all real quantities positive or negative are positive, these expressions cannot represent any *real* quantities and are hence called *imaginary quantities.*

The imaginary quantity $\sqrt{-1}$ is generally represented by the letter i.

9. Identities and Equations.

If the relation of equality between two algebraical expressions holds good for all values of the variables involved, then the relation is called an *Identity*. If, however, the equality holds good only for some value or values of the variables involved, then the relation is called an *Equation.*

Thus, the relation $(a+b)^2+(a-b)^2=2(a^2+b^2)$ is an identity, for it is true for all values of a and b; while the relation $x+4=3x-2$ is an equation, for it is true only when $x=3$, and not for all values of x.

When it is desired to point out particularly that the relation of equality is an identity and not merely an equation, the sign of equality is written $\equiv$. Thus the above identity may be written

$$(a+b)^2+(a-b)^2\equiv 2(a^2+b^2).$$

Usually however the ordinary sign of equality($=$) is used.

10. Roots of Equations.

The values of the variable (or variables) which satisfy the equation (or equations) are called the *roots* of the equation (or equations). Thus, the root of the equation $x+4=3x-2$ is $x=3$. Similarly, the roots of the simultaneous equations

$$\left.\begin{aligned} 3x+4y&=25 \\ 5x-3y&=3 \end{aligned}\right\}$$

are $x=3$, $y=4$; and of the simultaneous equations

$$\left.\begin{aligned} x+2y+3z&=14 \\ 2x+3y+z&=11 \\ 3x+y+2z&=11 \end{aligned}\right\}$$

are $x=1$, $y=2$, $z=3$.

In an equation involving only one variable, the number of its roots is equal to the degree of the equation.

Thus, a simple (*i.e.*, linear) equation has one root; a quadratic (*i.e.*, second degree) equation has two roots: a cubic (*i.e.*, third degree) equation has three roots: and in general, an equation of the nth degree has n roots. These roots however may be real or imaginary.

It follows that *if an equation of the nth degree in x has more than n roots, that is to say, is satisfied by more than n values of x, it is not really an equation but an identity.*

Thus, if a simple equation is satisfied by more than one value of x, or if a quadratic equation is satisfied by more than two values of x, and so on, then these equations are really identities, and as such are satisfied by all values of x.

This theorem may very usefully be applied to test whether an apparent equation is really so, or if it is really an identity.

11. Zero and Infinity.

A quantity which is numerically smaller than any assignable quantity however small is called *zero*, and is represented by the symbol 0; and a quantity which is numerically larger than any assignable quantity however large is called *infinity*, and is represented by the symbol ∞.

In short, an indefinitely small quantity is 0, and an indefinitely large quantity is ∞.

Zero can therefore be looked upon as the *limiting value* of a function which diminishes in such a manner as to become smaller than any assignable quantity however small. Corresponding remarks apply to infinity looked upon as a limiting value.

For example, consider the function $\frac{1}{x}$. As x becomes larger and larger, the function becomes smaller and smaller, so that when x becomes infinitely large the function becomes infinitely small. Hence, the limiting value of $\frac{1}{x}$ is 0, when x is ∞. Similarly, the limiting value of $\frac{1}{x}$ is ∞, when $x=0$.

From the above definitions of zero and infinity it is obvious that *all zeros are not necessarily equal nor are all infinities necessarily equal.* Consequently we are not justified in cancelling zero factors from the two sides of an identity or an equation.

It follows further that functions which, for some value or values of the variables, assume the forms

$$\frac{0}{0}, \quad 0\times\infty, \quad \infty-\infty, \quad \frac{\infty}{\infty}$$

are *indeterminate*.

Some important and useful results relating to zero and infinity are given below.

If a is a finite positive quantity,

$$\frac{a}{0}=\infty, \quad \frac{a}{\infty}=0; \quad \frac{0}{a}=0, \quad \frac{\infty}{a}=\infty; \quad 0\times a=0, \quad \infty\times a=\infty;$$

$$0^a=0, \quad \infty^a=\infty; \quad a^0=1. \quad a^\infty=\infty (\text{if } a>1), \; a^\infty=0 \; (\text{if } a<1).$$

12. Factor-Formulae.

We give below some important factor-formulæ :

$(a+b)^2=a^2+2ab+b^2.$

$(a-b)^2=a^2-2ab+b^2.$

$(a+b+c)^2=a^2+b^2+c^2+2bc+2ca+2ab.$

$(a+b+c+d)^2=a^2+b^2+c^2+d^2+2a(b+c+d)+2b(c+d)+2cd$

$(a+b)(a-b)=a^2-b^2.$

$(a^2+ab+b^2)(a^2-ab+b^2)=a^4+a^2b^2+b^4.$

$(x+a)(x+b)=x^2+(a+b)x+ab.$

$(x-a)(x-b)=x^2-(a+b)x+ab.$

$(x+a)(x-b)=x^2+(a-b)x-ab.$

$(x-a)(x+b)=x^2-(a-b)x-ab.$

$(px+q)(rx+s)=prx^2+(qr+ps)x+qs.$

$(x+a)(x+b)(x+c)=x^3+(a+b+c)x^2+(bc+ca+ab)x+abc.$

$(x+a)(x+b)(x-c)=x^3+(a+b-c)x^2+(-bc-ca+ab)x-abc.$

$(x+a)(x-b)(x-c)=x^3+(a-b-c)x^2+(bc-ca-ab)x+abc.$

$(x-a)(x-b)(x-c)=x^3-(a+b+c)x^2+(bc+ca+ab)x-abc.$

$(x+a)(x+b)(x+c)(x+d)$......to n factors

$$=x^n+s_1x^{n-1}+s_2x^{n-2}+s_3x^{n-3}+\ldots\ldots+s_{n-1}x+s_n.$$

$(a+b)^3=a^3+3a^2b+3ab^2+b^3=a^3+b^3+3ab(a+b).$

$(a-b)^3=a^3-3a^2b+3ab^2-b^3=a^3-b^3-3ab(a-b).$

$(a+b)(a^2-ab+b^2)=a^3+b^3.$

$(a-b)(a^2+ab+b^2)=a^3-b^3.$

$$\begin{aligned}\tfrac{1}{2}(a+b+c)\{(b-c)^2+(c-a)^2+(a-b)^2\}&=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)\\&=a^3+b^3+c^3-3abc.\end{aligned}$$

$$\begin{aligned}-(b-c)(c-a)(a-b)&=a^2(b-c)+b^2(c-a)+c^2(a-b)\\&=bc(b-c)+ca(c-a)+ab(a-b)\\&=-\{a(b^2-c^2)+b(c^2-a^2)+c(a^2-b^2)\}.\end{aligned}$$

$$\begin{aligned}(b+c)(c+a)(a+b)&=a^2(b+c)+b^2(c+a)+c^2(a+b)+2abc\\&=bc(b+c)+ca(c+a)+ab(a+b)+2abc\\&=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+2abc.\end{aligned}$$

$$\begin{aligned}(a+b+c)(bc+ca+ab)&=a^2(b+c)+b^2(c+a)+c^2(a+b)+3abc\\&=bc(b+c)+ca(c+a)+ab(a+b)+3abc\\&=a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)+3abc\end{aligned}$$

$$\begin{aligned}&(a+b+c)^3\\&=a^3+b^3+c^3+3a^2b+3ab^2+3b^2c+3bc^2+3c^2a+3ca^2+6abc\\&=a^3+b^3+c^3+3(b+c)(c+a)(a+b).\end{aligned}$$

CHAPTER II

THEORY OF INDICES AND EXPONENTIAL EQUATIONS

I. Theory of Indices.

13. The Laws of Indices.

The student has learnt in Elementary Algebra that when the indices are all positive integers :

(i) $a^m \times a^n = a^{m+n}$.

(ii) $a^m \times a^n \times a^p \times \cdots = a^{m+n+p+\cdots}$.

(iii) $(a^m)^n = a^{mn}$.

(iv) $a^m \div a^n = a^{m-n}$.

These are the Laws of Indices.

Of these, the first one is the fundamental Index Law, and the others follow from it.

14. Generalization of the Laws.

When the indices are positive integers, the meaning of the powers is evident; thus, when m is a positive integer, a^m means that a is multiplied by itself m times.

But this meaning cannot evidently hold good when the index is not a positive integer but is fractional or negative.

We have therefore to see if some meaning or interpretation can be found of quantities with fractional or negative indices, so that the laws of indices may hold good generally for all values of the indices. Assuming, then, the universal truth of the fundamental Index Law

$$a^m \times a^n = a^{m+n},$$

we proceed to give the interpretation.

15. The meaning of $a^{\frac{p}{q}}$, where p and q are positive integers.

Since, by our assumption, for all values of m and n,

$$a^m \times a^n = a^{m+n},$$

we have $$a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{p}{q}+\frac{p}{q}} = a^{\frac{2p}{q}},$$

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{p}{q}+\frac{p}{q}+\frac{p}{q}} = a^{\frac{3p}{q}},$$

Hence, $$a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \cdots \text{to } q \text{ factors}$$

$$= a^{\frac{p}{q}+\frac{p}{q}+\cdots\text{to } q \text{ terms}} = a^{q\cdot\frac{p}{q}} = a^p.$$

That is, $$\left(a^{\frac{p}{q}}\right)^q = a^p$$

$\therefore$ $$a^{\frac{p}{q}} = \sqrt[q]{a^p}, \text{ the } q^{th} \text{ root of } a^p.$$

Thus, $$a^{\frac{1}{2}} = \sqrt{a},\ a^{\frac{2}{3}} = \sqrt[3]{a^2},\ a^{\frac{3}{5}} = \sqrt[5]{a^3}, \text{ and so on.}$$

Again, if $p=1$, we have

$$a^{\frac{1}{q}} \times a^{\frac{1}{q}} \times a^{\frac{1}{q}} \times \cdots \text{to } q \text{ factors} = a^{q\cdot\frac{1}{q}} = a.$$

That is, $$(a^{\frac{1}{q}})^q = a.$$

$\therefore$ $$a^{\frac{1}{q}} = \sqrt[q]{a}, \text{ the } q^{th} \text{ root of } a.$$

Also, since $a^{\frac{1}{q}} \times a^{\frac{1}{q}} \times \cdots \cdots$ to p factors $= a^{\frac{p}{q}}$, we may also say that $a^{\frac{p}{q}}$ is equal to the p^{th} power of $a^{\frac{1}{q}}$ *i. e.*, of the q^{th} root of a.

Thus $a^{\frac{p}{q}}$ is the same as $\sqrt[q]{a^p}$ or $(\sqrt[q]{a})^p$.

Hence we have the following meaning for a positive fractional index:

When the index is a positive fraction, the numerator denotes a power and the denominator a root to be taken, of the base.

16. The meaning of a^{-n}, where n is a positive quantity.

By the Index Law, we have

$$a^m \times a^n \times a^{-n} = a^{m+n-n} = a^m.$$

Dividing by a^m, $a^n \times a^{-n} = 1$,

$$\therefore \qquad a^{-n} = \frac{1}{a^n}.$$

Thus, $a^{-3} = \frac{1}{a^3}$, $a^{-\frac{2}{5}} = \frac{1}{a^{\frac{2}{5}}} = \frac{1}{\sqrt[5]{a^2}}$

and in general, $\quad a^{-\frac{p}{q}} = \frac{1}{a^{\frac{p}{q}}} = \frac{1}{\sqrt[q]{a^p}}.$

Thus we have the following meaning for a negative index:

When the index is a negative quantity, it means that the reciprocal of the base with the same positive index is to be taken.

Cor. 1. $a^m \div a^n = a^{m-n}$, *for all values of m and n.*

For, $a^m \div a^n = a^m \times \frac{1}{a^n} = a^m \times a^{-n} = a^{m-n}$.

Cor. 2. The meaning of a^0.

By the Index Law, we have

$$a^n \times a^{-n} = a^{n-n} = a^0.$$

Also, $\qquad a^n \times a^{-n} = a^n \times \frac{1}{a^n} = 1$;

$$\therefore \qquad a^0 = 1.$$

That is, *when any finite quantity is raised to the power zero, the result is unity.*

17. To prove that $(a^m)^n = a^{mn}$, for all values of m and n.

(*i*) Let n be a positive integer.

Then whatever may be the value of m, we have, by the Index Law

$$\begin{aligned}(a^m)^n &= a^m \times a^m \times a^m \times \cdots \text{to } n \text{ factors}\\ &= a^{m+m+m+\ldots \text{to } n \text{ terms}}\\ &= a^{mn}.\end{aligned}$$

(*ii*) Let n be a positive fraction and equal to $\frac{p}{q}$.

Then whatever may be the value of m, we have

$$(a^m)^n=(a^m)^{\frac{p}{q}}=\sqrt[q]{(a^m)^p}=\sqrt[q]{a^{mp}}=a^{\frac{mp}{q}}=a^{mn}.$$

(*iii*) Let n be a negative quantity and equal to $-p$, where p is a positive quantity, integral or fractional; then whatever may be the value of m, we have

$$(a^m)^n=(a^m)^{-p}=\frac{1}{(a^m)^p}=\frac{1}{a^{mp}}=a^{-mp}=a^{mn}.$$

Thus, for all values of m and n, $(a^m)^n=a^{mn}$.

18. To prove that $(ab)^n=a^nb^n$ for all values of n.

(*i*) When n is a positive integer,

$(ab)^n=ab.\ ab.\ ab.\ \ldots\ldots$to n factors

$=(a.\ a.\ a.\cdots$to n factors$)\times(b.\ b.\ b.\ldots$to n factors$)$

$=a^nb^n$.

(*ii*) When n is a positive fraction, say $\frac{p}{q}$, where p and q are any two positive integers.

$$(ab)^n=(ab)^{\frac{p}{q}}=\sqrt[q]{(ab)^p}=\sqrt[q]{a^pb^p}=\sqrt[q]{a^{nq}b^{nq}}$$
$$=\sqrt[q]{(a^nb^n)^q}=(a^nb^n)^{\frac{q}{q}}=a^nb^n.$$

(*iii*) When n is a negative quantity, say, $-p$, so that p is a positive quantity, integral or fractional.

$$(ab)^n=(ab)^{-p}=\frac{1}{(ab)^p}=\frac{1}{a^pb^p}=a^{-p}b^{-p}=a^nb^n.$$

Cor. 1. $(abcd\cdots)^n=a^nb^nc^nd^n\ldots$, for all values of n.

For, $(abcd\ldots)^n=a^n(bcd\ldots)^n=a^nb^n(cd\ldots)^n$, and so on.

Cor. 2. $\left(\frac{a}{b}\right)^n=\left(a.\frac{1}{b}\right)^n=(a.b^{-1})^n=a^n(b^{-1})^n=a^nb^{-n}=\frac{a^n}{b^n}$.

Cor. 3. $\left(\frac{a}{b}\right)^{-n}=\frac{1}{\left(\frac{a}{b}\right)^n}=\frac{1}{\frac{a^n}{b^n}}=\frac{b^n}{a^n}=\left(\frac{b}{a}\right)^n$.

Example 1. *Find the value of* (*i*) $16^{\frac{5}{4}}$ *and* (*ii*) $9^{-\frac{3}{2}}$

(*i*) $16^{\frac{5}{4}}=(\sqrt[4]{16})^5=2^5=32.$

(*ii*) $9^{-\frac{3}{2}}=\frac{1}{9^{\frac{3}{2}}}=\frac{1}{(\sqrt{9})^3}=\frac{1}{3^3}=\frac{1}{27}.$

Example 2. *Find the value of* $\{(\sqrt[5]{125})^{-2}\}^{-\frac{5}{3}}$.

$$\left\{\left(\sqrt[5]{125}\right)^{-2}\right\}^{-\frac{5}{3}}=\left(125^{-\frac{2}{5}}\right)^{-\frac{5}{3}}=\left(125\right)^{\left(-\frac{2}{5}\right)\left(-\frac{5}{3}\right)}=125^{\frac{2}{3}}$$

$$=(\sqrt[3]{125})^2=5^2=25.$$

Example 3. *Simplify* $\left(\frac{x^m}{x^n}\right)^{m+n}\times\left(\frac{x^n}{x^l}\right)^{n+l}\times\left(\frac{x^l}{x^m}\right)^{l+m}$.

[*Cal.* 1900 & *Dac.* 1925]

The given expression $=(x^{m-n})^{m+n}\times(x^{n-l})^{n+l}\times(x^{l-m})^{l+m}$

$$=x^{m^2-n^2}\times x^{n^2-l^2}\times x^{l^2-m^2}$$

$$=x^{(m^2-n^2)+(n^2-l^2)+(l^2-m^2)}$$

$$=x^0$$

$$=1.$$

Example 4. *Divide* $x^{2^n}-y^{2^n}$ by $x^{2^{n-1}}+y^{2^{n-1}}$. [*Cal.* 1879]

Let $2^{n-1}=p$; then, $2^n=2^{n-1}.2=2p.$

$\therefore$ $$x^{2^n}-y^{2^n}=x^{2p}-y^{2p}=(x^p)^2-(y^p)^2$$

$$=(x^p+y^p)(x^p-y^p)$$

$$=\left(x^{2^{n-1}}+y^{2^{n-1}}\right)\left(x^{2^{n-1}}-y^{2^{n-1}}\right)$$

$\therefore$ the required quotient $=x^{2^{n-1}}-y^{2^{n-1}}$.

Example 5. *Simplify*

$$\frac{1}{1+x^{n-m}+x^{p-m}}+\frac{1}{1+x^{m-n}+x^{p-n}}+\frac{1}{1+x^{m-p}+x^{n-p}}$$

[*Dac.* 1929]

The 1st term $=\dfrac{x^m}{x^m(1+x^{n-m}+x^{p-m})}=\dfrac{x^m}{x^m+x^n+x^p}$;

the 2nd term $=\dfrac{x^n}{x^n(1+x^{m-n}+x^{p-n})}=\dfrac{x^n}{x^n+x^m+x^p}$;

and the 3rd term $=\dfrac{x^p}{x^p(1+x^{m-p}+x^{n-p})}=\dfrac{x^p}{x^p+x^m+x^n}$.

Hence by addition, the given expression

$$=\frac{x^m}{x^m+x^n+x^p}+\frac{x^n}{x^n+x^m+x^p}+\frac{x^p}{x^p+x^m+x^n}$$

$$=\frac{x^m+x^n+x^p}{x^m+x^n+x^p}$$

$$=1.$$

EXERCISE 1

1. Express the following with positive indices :

$3x^{-\frac{2}{5}}$; $5x^{-\frac{1}{3}}a^{-\frac{2}{3}}$; $\dfrac{1}{3x^{-\frac{2}{3}}}$; $x^{-\frac{3}{4}}+a^{-\frac{4}{3}}$; and $a^{-3}x^{-2}\div a^{-2}x^{-m}$.

2. Express the following with radical signs avoiding fractions and negative indices :

$$a^{\frac{3}{4}};\ x^{-\frac{1}{4}};\ x^{-\frac{3}{5}};\ 2a^{-\frac{1}{x}}\,b^{-\frac{1}{y}};\ \frac{x^{-\frac{2}{3}}}{y^{-\frac{3}{4}}};\ a^{-\frac{x}{y}}\div b^{-\frac{y}{x}}.$$

3. Express the following with a single positive index :

$$\left(a^{-\frac{1}{2}}\right)^{\frac{2}{3}};\ \left(a^{2}\right)^{\frac{3}{4}};\ \left(a^{-2}\right)^{-\frac{1}{2}};\ \left(x^{\frac{2}{3}}\right)^{-\frac{3}{4}};\ \left(x^{m^{n-1}}\right)^{m};\ \left(a^{\sqrt[3]{2}}\right)^{\sqrt[3]{4}}.$$

4. Find the value of :

$$4^{\frac{3}{2}};\ 4^{-\frac{1}{2}};\ 8^{-\frac{4}{3}};\ 27^{\frac{4}{3}};\ 625^{-\frac{3}{4}};\ \left(\frac{1}{9}\right)^{-\frac{3}{2}};\ \left(\frac{1}{81}\right)^{-\frac{3}{4}};\ \left(\frac{8}{125}\right)^{\frac{4}{3}}.$$

5. Simplify :

(i) $\left\{\sqrt[3]{x^{-\frac{3}{2}}}\right\}^{-\frac{2}{3}}$. (ii) $\left\{\left(\sqrt[3]{x^4}\right)^{-\frac{3}{2}}\right\}^{-\frac{1}{2}}$.

(iii) $\left(a^{-\frac{1}{2}}\right)^2 \times \left(b^{\frac{1}{3}}\right)^{-\frac{1}{3}}$. (iv) $\sqrt[3]{a^{-2}} \div \sqrt[4]{a^{-3}}$.

(v) $\sqrt{a^2 b^{-\frac{9}{4}}} \times \sqrt[4]{a^{-\frac{8}{3}} b^3}$. (vi) $\left(x^{\frac{1}{4}} y^{-\frac{1}{2}}\right)^{\frac{4}{3}} \div \left(xy^{-1}\right)^{\frac{1}{3}}$.

(vii) $\dfrac{x^{a+b} . x^{a-b} . x^{m-2a}}{x^{m-a}}$. [Cal. 1870]

(viii) $\dfrac{x^{m+2n} . x^{3m-8n}}{x^{5m-6n}}$. [Cal. 1874]

(ix) $\dfrac{x^{b+c}}{x^a} . \dfrac{x^{c+a}}{x^b} . \dfrac{x^{a+b}}{x^c}$. (x) $\left(\dfrac{a^l}{a^m}\right)^n \left(\dfrac{b^n}{c^l}\right)^m \left(\dfrac{c^m}{a^n}\right)^l$.

(xi) $\left(x^{\frac{a}{a-b}}\right)^{\frac{a}{a-c}} \left(x^{\frac{b}{b-c}}\right)^{\frac{b}{b-a}} \left(x^{\frac{c}{c-a}}\right)^{\frac{c}{c-b}}$. (xii) $\left\{\left(a^{\frac{1}{m+1}}\right)^{m-\frac{1}{m}}\right\}^{\frac{1}{m-1}}$.

(xiii) $\left(\dfrac{x^p}{x^q}\right)^{p+q} \div \left(\dfrac{x^{p+q}}{x^{p-q}}\right)^{\frac{p}{q}^{2}}$ [Cal. 1902]

(xiv) $\sqrt[bc]{\dfrac{x^b}{x^c}} . \sqrt[ca]{\dfrac{x^c}{x^a}} . \sqrt[ab]{\dfrac{x^a}{x^b}}$. [Dac. 1928]

(xv) $\dfrac{\left(p+\dfrac{1}{q}\right)^m \left(p-\dfrac{1}{q}\right)^m}{\left(q+\dfrac{1}{p}\right)^m \left(q-\dfrac{1}{p}\right)^m}$ [Bom. 1889]

(xvi) $\dfrac{\left(p^2-\dfrac{1}{q^2}\right)^p \left(p-\dfrac{1}{q}\right)^{q-p}}{\left(q^2-\dfrac{1}{p^2}\right)^q \left(q+\dfrac{1}{p}\right)^{p-q}}$, (Bom. 1891)

(xvii) $\dfrac{\left(1+\dfrac{a}{b}\right)^{\frac{a}{a-b}} \left(1-\dfrac{b}{a}\right)^{\frac{b}{a-b}}}{\left(\dfrac{b}{a}+1\right)^{\frac{a}{a-b}} \left(\dfrac{a}{b}-1\right)^{\frac{b}{a-b}}}$ (xviii) $\dfrac{x^{\frac{2}{3}}-y^{\frac{2}{3}}}{x^{\frac{2}{3}}+y^{\frac{2}{3}}} \div \dfrac{x-x^{\frac{1}{3}}y^{\frac{2}{3}}}{x^{\frac{2}{3}}y^{\frac{1}{3}}+y}$.

(xix) $\dfrac{1}{1+a^{-m}b^n+a^{-m}c^p} + \dfrac{1}{1+b^{-n}c^p+b^{-n}a^m} + \dfrac{1}{1+c^{-p}a^m+c^{-p}b^n}$

(xx) $\left(\dfrac{x^l}{x^m}\right)^{l^2+lm+m^2} \times \left(\dfrac{x^m}{x^n}\right)^{m^2+mn+n^2} \times \left(\dfrac{x^n}{x^l}\right)^{n^2+nl+l^2}$ [Cal. 1904]

6. Multiply $x^{2^{n-1}} + y^{2^{n-1}}$ by $x^{2^{n-1}} - y^{2^{n-1}}$.

7. Multiply $x^{2^{n-1}} + a^{2^{n-2}} x^{2^{n-2}} + a^{2^{n-1}}$

by $x^{2^{n-1}} - a^{2^{n-2}} x^{2^{n-2}} + a^{2^{n-1}}$.

8. Divide $x^{2^n}+a^{2^{n-1}}x^{2^{n-1}}+a^{2^n}$ by $x^{2^{n-1}}-a^{2^{n-2}}x^{2^{n-2}}+a^{2^{n-1}}$.

9. Divide $a^{3^n}+b^{3^n}+c^{3^n}-3a^{3^{n-1}}b^{3^{n-1}}c^{3^{n-1}}$ by $a^{3^{n-1}}+b^{3^{n-1}}+c^{3^{n-1}}$.

10. If $m=a^{\frac{1}{3}}+a^{-\frac{1}{3}}$, show that $m^3-3m=a+\frac{1}{a}$.

11. If $x=(r+\sqrt{r^2+q^3})^{\frac{1}{3}}+(r-\sqrt{r^2+q^3})^{\frac{1}{3}}$, find the value of $x^3+3qx-2r$. [Mad. F. A. 1885]

12. Find the square of $x^{\frac{1}{2}}-1+x^{-\frac{1}{2}}$.

13. Divide $x^2y^{-2}+x^{-2}y^2+2$ by $x^{\frac{2}{3}}y^{-\frac{2}{3}}+x^{-\frac{2}{3}}y^{\frac{2}{3}}-1$. [Mad. F. A. 1894]

14. Divide $ab^{-1}+bc^{-1}+ca^{-1}-3$ by $a^{\frac{1}{3}}b^{-\frac{1}{3}}+b^{\frac{1}{3}}c^{-\frac{1}{3}}+c^{\frac{1}{3}}a^{-\frac{1}{3}}$

15. Show that

$$a^3b^{-3}-a^{-3}b^3=(ab^{-1}-a^{-1}b)(ab^{-1}+a^{-1}b+1)(ab^{-1}+a^{-1}b-1).$$

16. Show that

$$\frac{x^{2^n}-a^{2^n}}{x-a}=(x+a)(x^2+a^2)(x^4+a^4)\cdots\left(x^{2^{n-1}}+a^{2^{n-1}}\right).$$

17. If $\left(\frac{b}{c}\right)^l\left(\frac{c}{a}\right)^m\left(\frac{a}{b}\right)^n=1$, prove that

$$\left(\frac{b}{c}\right)^{\frac{1}{m-n}}=\left(\frac{c}{a}\right)^{\frac{1}{n-l}}=\left(\frac{a}{b}\right)^{\frac{1}{l-m}}.$$

II. Exponential Equations.

19. Definition and Method of Solution.

Equations in which the unknown quantities appear as indices or exponents are called *exponential equations.*

Thus, $2^{3x}=4^{x+1}$ is an exponential equation; and $2^x+3^y=4$, $2^{x+1}+3^{y+1}=11$ are simultaneous exponential equations.

In solving exponential equations, we use the following

Axiom. *If the same base raised to certain powers gives equal results, then the indices of the powers must be equal.*

Thus, if $a^x=a^y=a^z$, then $x=y=z$; similarly, if $2^x=4^y=(2^2)^y=2^{2y}$, then $x=2y$; and so on.

The method of solving exponential equations is illustrated in the following examples.

Example 1. *Solve* $(\sqrt[3]{2})^{2x+7}=(\sqrt[4]{2})^{7x+5}$

From the given equation, we have

$$2^{\frac{2x+7}{3}}=2^{\frac{7x+5}{4}}$$

$$\therefore \quad \frac{2x+7}{3}=\frac{7x+5}{4} \qquad \text{[by the Axiom]}$$

$$\text{or,} \quad 4(2x+7)=3(7x+5)$$

$$\text{or} \quad 8x+28=21x+15$$

$$\text{or,} \quad 13x=13$$

$$\therefore \quad x=1.$$

Example 2. *Solve* $a^{x+y}=(\sqrt{a})^{y+7}$ (1)

$$b^{x-y}=\left(\frac{1}{\sqrt[4]{b}}\right)^{y+1} \qquad (2)$$

From (1), we have $a^{x+y}=a^{\frac{y+7}{2}}$

$$\therefore \quad x+y=\frac{y+7}{2} \qquad \text{(by the Axiom)}$$

$$\text{or,} \quad 2x+2y=y+7$$

$$\text{or,} \quad 2x+y-7=0 \qquad \cdots \qquad (3)$$

From (2), we have, $b^{x-y}=b^{-\frac{y+1}{4}}$

$$x-y=-\frac{y+1}{4} \qquad \text{[by the Axiom]}$$

$$\text{or,} \quad 4x-4y=-y-1$$

$$\therefore \quad 4x-3y+1=0 \qquad \cdots \qquad \cdots \qquad (4)$$

From (3) and (4), by cross-multiplication,

$$\frac{x}{1-21}=\frac{y}{-28-2}=\frac{1}{-6-4}, \quad \text{or,} \quad \frac{x}{20}=\frac{y}{30}=\frac{1}{10}.$$

$$\therefore \quad x=2, \text{ and } y=3.$$

EXERCISE 2.

Solve the following equations :

1. $3^{2x+1} = (\sqrt{3})^{x+3}$.
2. $3^{2x+3} = 5^{2x+3}$.
3. $3^{x+2} - 3^x = 72$.
4. $(\sqrt{5})^{4x-4} - 5^{2x-3} = 20$.
5. $m^{ax+b} = 1$.
6. $a^x = \{(\sqrt{a})^{3x}\}^x$.
7. $(\sqrt{6})^{2x+4} = 216$ [Cal. 1932]
8. $4.2^{x-1} = 8^x$. [Dac. 1931]
9. $(\sqrt[5]{9})^{4x+3} = (\sqrt[7]{3})^{2x+49}$. [Dac. 1935]
10. $9^x = \dfrac{9}{3^x}$. [Pun. 1930]
11. $a^{2x^m} = \left\{(a^x)^m\right\}^x$.
12. $a^{-x}(a^x + b^{-x}) = \dfrac{a^2b^2+1}{a^2b^2}$.
13. $\left(\dfrac{a}{b}\right)^{px+q} = \left(\dfrac{b}{a}\right)^{qx-p}$.
14. $x^y = y^x$, $x = 2y$. [Cal. Int. 1935]
15. $x^y = y^x$, $x^b = y^a$.
16. $a^x = x^y$, $a^y = x^x$.
17. $2^x + 3^y = 7$, $2^x - 3^y = 1$.
18. $2^x + 3^y = 7$, $4^x - 9^y = 7$.
19. $2^{2x-1} = 32^{y-1}$, $(\sqrt[3]{3})^x = 3^{5-2y}$.
20. $a^x + b^y = a + b$, $a^{x-1} + b^{y-1} = 2$.
21. $a^x . a^{y+1} = a^7$, $a^{2y} . a^{3x+5} = a^{20}$. [Cal. 1879]
22. $36^x = 6^{z+x}$, $16^{x-y} = 4$, $9^{x+y+z} = 243$.
23. $a^x = (x+y+z)^y$, $a^y = (x+y+z)^z$, $a^z = (x+y+z)^x$.
24. $(xyz)^{x^2} = a^{3yz}$, $(xyz)^{y^2} = a^{3zx}$, $(xyz)^{z^2} = a^{3xy}$.
25. If $xy^{p-1} = a$, $xy^{q-1} = b$, $xy^{r-1} = c$, show that $a^{q-r}\, b^{r-p}\, c^{p-q} = 1$. [Pat. 1920]
26. If $x^y = y^x$, show that $\left(\dfrac{x}{y}\right)^{\frac{x}{y}} = x^{\frac{x}{y}-1}$; and if $x = 2y$, prove that $y = 2$. [Cal. 1928]
27. If $a^{m^n} = (a^m)^n$, find m in terms of n. [Pat. 1918]
28. Eliminate y from $m = y^x$ and $n = x^y$. [Pun. 1929]
29. If $x = 3^{\frac{1}{3}} - 3^{-\frac{1}{3}}$, prove that $3x^3 + 9x = 8$. [Pat. 1930]
30. If $m = a^x$, $n = a^y$, and $a^2 = (m^y n^x)^z$, show that $xyz = 1$. [Bom. 1890, & Pat. 1919 & 1921]

CHAPTER III

SQUARE ROOTS

20. Definition.

The *square root* of a given quantity is that quantity which when multiplied by itself produces the given quantity. Thus, since

$$(x+y)(x+y)=(x+y)^2=x^2+2xy+y^2,$$

and $(-x-y)(-x-y)=\{-(x+y)\}^2=x^2+2xy+y^2$, both $x+y$ and $-x-y$ are square roots of $x^2+2xy+y^2$.

It is evident that any positive quantity has two square roots which are numerically equal but opposite in sign.

Thus, the square roots of $x^2+2xy+y^2$ are $\pm(x+y)$, those of b^2 are $\pm b$, those of a (a being positive) are $\pm\sqrt{a}$, and so on.

This double sign in the square root is expressed by the sign $\pm$, as shown above; but usually in extracting square roots the sign is not actually written, the double sign being understood.

A negative quantity can have no *real* square roots; its square roots can only be *imaginary* quantities. [See Art. 8.]

21. Square roots of Monomials.

Since we have, by the Theory of Indices,

$$\sqrt{abc\cdots}=(abc\cdots)^{\frac{1}{2}}=a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}\cdots\cdots=\sqrt{a}\sqrt{b}\sqrt{c}\cdots\cdots,$$

we see that the square root of any monomial consisting of any number of factors is equal to the product of the square roots of the factors taken separately.

22. Square roots of Polynomials.

From the formula

$$(a+b)^2=a^2+2ab+b^2=a^2+b(2a+b),$$

we may deduce a method for the extraction of the square root. The process is actually arranged thus :

$$\begin{array}{r|l} & a^2+2ab+b^2\,(a+b \\ & \underline{a^2} \\ 2a+b & 2ab+b^2 \\ & \underline{2ab+b^2} \end{array}$$

Thus we get *the process of finding out the square root of any polynomial* :

(i) Arrange the expression according to ascending or descending powers of some common letter.

(ii) Take the square root of the first term of the expression for the first term of the required root; subtract its square from the given expression and bring down the remainder.

(iii) Divide the first term of the remainder by twice the first term of the root and take the quotient thus obtained for the second term of the root, and also add it to twice the first term to obtain the complete divisor of the remainder.

(iv) Multiply the divisor thus obtained by the second term of the root and subtract the product from the first remainder, as in division.

(v) Considering the two terms of the root thus obtained as one term, continue the process as before until there is no remainder.

Apparently different square roots may be obtained by arranging the given expression in ascending or descending powers of different letters; but it will always be found that these roots are either identical or one of the roots is the other with the sign changed. [Art. 20.]

The following examples will illustrate the method.

Example 1. *Extract the square root of*

$$4x^4-12x^3+29x^2-30x+25.$$

$$\begin{array}{l|l|l}
 & 4x^4-12x^3+29x^2-30x+25 & 2x^2-3x+5 \\
 & 4x^4 & \\
\hline
4x^2-3x & -12x^3+29x^2-30x+25 & \\
 & -12x^3+9x^2 & \\
\hline
4x^2-6x+5 & 20x^2-30x+25 & \\
 & 20x^2-30x+25 & \\
\hline
\end{array}$$

Hence the required square root $=2x^2-3x+5$.

Example 2. *Find the square root of*

$$a^2x^4+2abx^3+b^2x^2+2cax^2+2bcx+c^2.$$

Arranging the terms in descending powers of x, we have

$$\begin{array}{l|l|l}
 & a^2x^4+2abx^3+(b^2+2ca)x^2+2bcx+c^2 & ax^2+bx+c \\
 & a^2x^4 & \\
\hline
2ax^2+bx & 2abx^3+(b^2+2ca)x^2+2bcx+c^2 & \\
 & 2abx^3+b^2x^2 & \\
\hline
2ax^2+2bx+c & 2cax^2+2bcx+c^2 & \\
 & 2cax^2+2bcx+c^2 & \\
\hline
\end{array}$$

Thus the required square root $=ax^2+bx+c$.

Example 3. *Find the square root of* $x^2+4x+10+\frac{12}{x}+\frac{9}{x^2}$.

[*Mad.* 1880]

Here the expression is arranged according to descending powers of x, namely, x^2, x, $x^0(=1)$, $\frac{1}{x}\left(=x^{-1}\right)$, and $\frac{1}{x^2}\left(=x^{-2}\right)$, the indices of x being 2, 1, 0, -1 and -2 respectively.

$$
\begin{array}{l}
\;x^2+4x+10+\frac{12}{x}+\frac{9}{x^2}\;\Big(x+2+\frac{3}{x}\\
\;x^2\\
\hline
2x+2\,\big)\;4x+10\\
\;4x+4\\
\hline
2x+4+\frac{3}{x}\,\big)\;6+\frac{12}{x}+\frac{9}{x^2}\\
\phantom{2x+4+\frac{3}{x})}\;6+\frac{12}{x}+\frac{9}{x^2}
\end{array}
$$

Thus the required square root $=x+2+\frac{3}{x}$.

Example 4. *Find the square root of* $x^{\frac{4}{3}}+10x^{\frac{2}{3}}-4x-12x^{\frac{1}{3}}+9$.

Arrange the expression according to descending powers of x, and proceed thus :

$$
\begin{array}{l}
\phantom{2x^{\frac{2}{3}}-2x^{\frac{1}{3}})}\;x^{\frac{4}{3}}-4x+10x^{\frac{2}{3}}-12x^{\frac{1}{3}}+9\;\big(x^{\frac{2}{3}}-2x^{\frac{1}{3}}+3\\
\phantom{2x^{\frac{2}{3}}-2x^{\frac{1}{3}})}\;x^{\frac{4}{3}}\\
\hline
2x^{\frac{2}{3}}-2x^{\frac{1}{3}}\,\big)-4x+10x^{\frac{2}{3}}\\
\phantom{2x^{\frac{2}{3}}-2x^{\frac{1}{3}})}-4x+4x^{\frac{2}{3}}\\
\hline
2x^{\frac{2}{3}}-4x^{\frac{1}{3}}+3\,\big)6x^{\frac{2}{3}}-12x^{\frac{1}{3}}+9\\
\phantom{2x^{\frac{2}{3}}-4x^{\frac{1}{3}}+3)}6x^{\frac{2}{3}}-12x^{\frac{1}{3}}+9\\
\hline
\end{array}
$$

Thus the required square root $=x^{\frac{2}{3}}-2x^{\frac{1}{3}}+3$.

23. Approximate Square Roots.

Sometimes we come across expressions whose square roots cannot be found *exactly*. In such cases we may find the square root approximately up to any desired degree of accuracy. An example is shown below.

Example 1. *Find the first four terms of the square root of* a^2+x^2. [*Cal.* 1877]

$$
\begin{array}{rl}
 & a^2+x^2\left(a+\dfrac{x^2}{2a}-\dfrac{x^4}{8a^3}+\dfrac{x^6}{16a^5}\right. \\
 & a^2 \\
2a+\dfrac{x^2}{2a}\Big) & x^2 \\
 & x^2+\dfrac{x^4}{4a^2} \\
2a+\dfrac{x^2}{a}-\dfrac{x^4}{8a^3}\Big) & -\dfrac{x^4}{4a^2} \\
 & -\dfrac{x^4}{4a^2}-\dfrac{x^6}{8a^4}+\dfrac{x^8}{64a^6} \\
2a+\dfrac{x^2}{a}-\dfrac{x^4}{4a^3}+\dfrac{x^6}{16a^5}\Big) & \dfrac{x^6}{8a^4}-\dfrac{x^8}{64a^6} \\
 & \dfrac{x^6}{8a^4}+\dfrac{x^8}{16a^6}-\dfrac{x^{10}}{64a^8}+\dfrac{x^{12}}{256a^{10}} \\
 & -\dfrac{5x^8}{64a^6}+\dfrac{x^{10}}{64a^8}-\dfrac{x^{12}}{256a^{10}}
\end{array}
$$

$\therefore$ the required square root $=a+\dfrac{x^2}{2a}-\dfrac{x^4}{8a^3}+\dfrac{x^6}{16a^5}$.

EXERCISE 3.

Find the square root of :

1. $9x^2-24xy+16y^2$.
2. $a^2+b^2+c^2-2ab-2bc+2ca$.
3. $x^4+4x^3+10x^2+12x+9$.
4. $4x^2+9y^2-12xy+16x-24y+16$.
5. $x^6+8x^4-2x^3+16x^2-8x+1$. [Cal. 1867]
6. $4x^4+8ax^3+4a^2x^2+16b^2x^2+16ab^2x+16b^4$. [Cal. 1870]
7. $1-4x+10x^2-20x^3+25x^4-24x^5+16x^6$ [All. 1893]
8. $4x^6-12x^5+13x^4-22x^3+25x^2-8x+16$. [Cal. 1893]
9. $\dfrac{x^2}{a^2}-\dfrac{2x}{a}+3-\dfrac{2a}{x}+\dfrac{a^2}{x^2}$. [Bom. 1890]
10. $\dfrac{x^4}{4}+4x^2+\dfrac{ax^2}{3}+\dfrac{a^2}{9}-2x^3-\dfrac{4ax}{3}$. [Cal. 1889]
11. $\left(a^2+\dfrac{1}{a^2}\right)-2\left(a+\dfrac{1}{a}\right)+3$. [All. 1904]

12. $\frac{1}{4}x^4-\frac{2}{3}x^3-\frac{11}{36}x^2+x+\frac{9}{16}$. [Mad. 1886]

13. $\frac{x^2}{y^2}+\frac{y^2}{x^2}-\frac{x}{y}+\frac{y}{x}-1\frac{3}{4}$. [Bom. 1893]

14. $25\frac{3}{7}-\frac{20x}{7y}+\frac{9y^2}{16x^2}-\frac{15y}{2x}+\frac{4x^2}{49y^2}$. [Dac. 1935]

15. $\left(\frac{a^4}{b^4}+\frac{b^4}{a^4}\right)-2\left(\frac{a^3}{b^3}+\frac{b^3}{a^3}\right)+\left(\frac{a^2}{b^2}+\frac{b^2}{a^2}\right)+2\left(\frac{a}{b}+\frac{b}{a}\right)-4$.

16. $x^4+4x^3(1-x)+4x(1-x)^3+(1-x)^4+6x^2(1-x)^2$.

17. $(a+b)^6-6ab(a+b)^4+9a^2b^2(a+b)^2-4a^3b^3$.

18. $x^{\frac{3}{2}}+9x^{-\frac{1}{2}}+6x^{\frac{1}{2}}+12x^{-\frac{1}{4}}+4x^{\frac{3}{4}}+4$. [Pat. 1935]

19. $a+b^{-1}+a^{-1}b+2b^{\frac{1}{2}}+2a^{-\frac{1}{2}}+2a^{\frac{1}{2}}b^{-\frac{1}{2}}$.

20. $a^{\frac{4m}{3n}}-a^{\frac{2m}{3n}}+1-2a^{\frac{m}{3n}}+2a^{\frac{m}{n}}$.

21. $4a-12a^{\frac{1}{2}}b^{\frac{1}{3}}+9b^{\frac{2}{3}}+16a^{\frac{1}{2}}c^{\frac{1}{4}}-24b^{\frac{1}{3}}c^{\frac{1}{4}}+16c^{\frac{1}{2}}$. [Bom. 1868]

22. $(x^{\frac{1}{3}}-y^{\frac{1}{3}})(x^{\frac{1}{3}}y^{\frac{2}{3}}-y+2x^{\frac{2}{3}}y-\frac{1}{6})+x^{\frac{4}{3}}y+x^{\frac{4}{3}}y^{-1}$.
[Mad. F. A. 1893]

23. $2x^{\frac{1}{3}}(3x+4)+2x^{-1}(x^{\frac{1}{3}}+1)-2\sqrt{2}x^{-\frac{5}{6}}(3x+2)$.
[Mad. F. A. 1895]

24. Find the square root of each of the following to four terms :

(i) $1+x$. (ii) $1-x$. (iii) $1-2x$.

(iv) $a+x$. (v) $1+x+x^2$. (vi) $1-x-x^2$. [Cal. 1885]

24. Square roots found by Formulæ.

In some cases the square roots of polynomials may be conveniently found by reducing them to one or other of the forms :

$$a^2\pm 2ab+b^2=(a\pm b)^2.$$

The following examples will illustrate this.

Example 1. *Find the square root of* $(a^2+b^2)^2+4ab(a^2-b^2)$.

The given exp. $=a^4+2a^2b^2+b^4+4ab(a^2-b^2)$
$=a^4-2a^2b^2+b^4+4ab(a^2-b^2)+4a^2b^2$
$=(a^2-b^2)^2+2.2ab(a^2-b^2)+(2ab)^2$
$=(a^2-b^2+2ab)^2$.

$\therefore$ the reqd. sq. root $=a^2-b^2+2ab$.

Example 2. *Find the square root of*

$(2n-1)(2n-3)(2n-5)(2n-7)+16.$ [*Bom.* 1904]

The given exp. $=\{(2n-1)(2n-7)\}\{(2n-3)(2n-5)\}+16$

$=(4n^2-16n+7)(4n^2-16n+15)+16$

$=(x+7)(x+15)+16$ [writing x for $4n^2-16n$]

$=x^2+22x+121$

$=(x+11)^2$

$=(4n^2-16n+11)^2$ [writing $4n^2-16n$ for x]

$\therefore$ the reqd. sq. root $=4n^2-16n+11.$

Example 3. *Find the square root of* $x^2+\frac{1}{x^2}-4\left(x+\frac{1}{x}\right)+6.$

[*Pun.* 1908]

The given expression $=\left(x^2+2+\frac{1}{x^2}\right)-4\left(x+\frac{1}{x}\right)+4$

$=\left(x+\frac{1}{x}\right)^2-4\left(x+\frac{1}{x}\right)+4$

$=\left(x-2+\frac{1}{x}\right)^2$

$\therefore$ the reqd. sq. root $=x-2+\frac{1}{x}$.

Example 4. *Find the square root of* $(ab+ac+bc)^2-4abc(a+c).$

[*Cal.* 1888]

Let $ab+bc=x$, and $ac=y$; then the given expression

$=(ab+ac+bc)^2-4ac(ab+bc)$

$=(x+y)^2-4xy$

$=(x-y)^2$

$=(ab+bc-ac)^2.$

$\therefore$ the reqd. sq. root $=ab+bc-ac.$

25. Condition for a Perfect Square

It may be required to find out the condition that must be satisfied in order that an expression may be a perfect square. The method will be evident from the example shown below.

Example 1. *Find the condition that ax^2+bx+c may be a perfect square.*

$$\begin{array}{r|l}
 & ax^2+bx+c\ \left(\sqrt{a}.x+\dfrac{b}{2\sqrt{a}}\right. \\
 & ax^2 \\ \hline
2\sqrt{a}.x+\dfrac{b}{2\sqrt{a}} & \quad bx+c \\
 & \quad bx+\dfrac{b^2}{4a} \\ \hline
 & \quad c-\dfrac{b^2}{4a}
\end{array}$$

Hence in order that ax^2+bx+c may be a perfect square we must have the last remainder not containing x equal to zero, *i.e.*, $c-\dfrac{b^2}{4a}=0$, or, $b^2=4ac$, which is the condition required.

EXERCISE 4.

Find the square root of :

1. $a^2+b^2+c^2-2ab+2ac-2bc$.
2. $4(x^2+1)^2+4(x^2+1)+1$.
3. $4-4c+2b+c^2-bc+\dfrac{b^2}{4}$. [Cal. 1876]
4. $(a-b)^4-2(a^2+b^2)(a-b)^2+2(a^4+b^4)$. [Bom. 1887]
5. $a^2+b^2+c^2+d^2-2a(b-c+d)-2b(c-d)-2cd$. [Cal. 1868]
6. $a^4+b^4+c^4+d^4-2(a^2+c^2)(b^2+d^2)+2a^2c^2+2b^2d^2$. [Cal. 1867]
7. $(x+3)(x+4)(x+5)(x+6)+1$.
8. $x(x-1)(x-2)(x-3)+1$.
9. $(x^2+3x+7)(x^2+5x+3)+(x-2)^2$.
10. $(x+a)(x+2a)(x+3a)(x+4a)+a^4$.
11. $\dfrac{x^2}{y^2}-\dfrac{2x}{y}+3-\dfrac{2y}{x}+\dfrac{y^2}{x^2}$. [Cal. 1909]
12. $\left(x+\dfrac{1}{x}\right)^2-4\left(x-\dfrac{1}{x}\right)$. [Cal. 1907]

13. $\left(x^2+\frac{1}{x^2}\right)^2-4\left(x+\frac{1}{x}\right)^2+12.$ [Cal. 1866]

14. $x^4+\frac{1}{x^4}+4\left(x+\frac{1}{x}\right)\left(x-\frac{1}{x}\right)+2.$ [Bom. 1928]

15. $2\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)+\frac{a^2}{b^2c^2}+\frac{b^2}{c^2a^2}+\frac{c^2}{a^2b^2}.$

16. $\frac{x^2}{y^2}\left(\frac{x^2}{4y^2}+1\right)+\frac{4y^2}{x^2}\left(\frac{y^2}{x^2}+1\right)+3.$

17. $9a-12a^{\frac{1}{2}}-2+4a^{-\frac{1}{2}}+a^{-1}.$ [Pat. 1930]

18. $\frac{(a^2+b^2)^2}{a^4+b^4-2a^2b^2}+4\frac{a}{a+b}\frac{b}{a-b}.$ [Cal. 1886]

19. $x^6+\frac{1}{x^6}+6\left(x^4+\frac{1}{x^4}\right)+15\left(x^2+\frac{1}{x^2}\right)+20.$ [Mad. 1899]

20. $\frac{x^4}{y^4}+\frac{y^4}{x^4}-2\left(\frac{x^3}{y^3}+\frac{y^3}{x^3}\right)+3\left(\frac{x^2}{y^2}+\frac{y^2}{x^2}\right)-4\left(\frac{x}{y}+\frac{y}{x}\right)+5.$ [Mad. 1882]

21. $(a^2+b^2)(a^2b^2+1)-2ab(a^2-1)(b^2-1)-4a^2b^2.$ [Mad. 1887]

22. $\frac{4x^2}{9y^2}-\frac{x}{z}-\frac{16x^2}{15yz}+\frac{9y^2}{16z^2}+\frac{6xy}{5z^2}+\frac{16x^2}{25z^2}.$ [Mad. 1883]

23. $a^4+\frac{1}{a^4}+a^2+\frac{1}{a^2}+\left\{a^3-\frac{1}{a^3}-\left(a-\frac{1}{a}\right)\right\}.$ [Mad. 1886]

24. $(m+n)^2+4ax\left[\frac{n^2}{(a-x)^2}-\frac{m^2}{(a+x)^2}\right].$

25. $x^2-2+\frac{4x^3+9x^2+4x+9}{x^4+4x^3+5x^2+4x+4}.$ [Mad. 1871]

26. Find the condition that x^2-px+q may be a perfect square.

27. Find a so that $4x^4+12x^3+25x^2+ax+16$ may be a perfect square. [Pat. 1932]

28. Find the value of a so that $4x+8x^{\frac{1}{2}}+4x^{-\frac{1}{2}}+4x^{-1}+x^{-2}+a$ may be a perfect square. [Pat. 1934]

29. Find the square root of $4x^4+8x^3+8x^2+4x+1$; hence find the square root of 48841. [Pun. 1932]

30. Show that $(a-b)^2(c+d)^2+4ab(c^2+d^2)-4cd(a^2+b^2)$ is an exact square. [Mad. 1873]

CHAPTER IV

SURDS

26. Definitions.

When any root of a number cannot be exactly obtained, the root is called a *surd* or an *irrational quantity*. Surds cannot therefore be exactly expressed by an integer or by a fraction or by a recurring or terminating decimal, but can only be expressed by a non-recurring and non-terminating decimal.

Thus $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{5}$, $4^{\frac{2}{3}}$, $7^{\frac{3}{4}}$, etc. are all surds, because their values cannot be exactly found. Their values can however be approximately determined to any desired degree of accuracy by proceeding to a sufficient number of decimal places.

Quantities which are not surds are called *rational* quantities. Thus 2, -3, $\frac{5}{6}$, 1·25, ·3 are all rational quantities. Also $\sqrt{4}$, $\sqrt{9}$, $\sqrt[3]{27}$, etc., though they are expressed in the surd form, are not really surds but are rational quantities, since these roots can be exactly obtained.

An algebraical quantity, such as $\sqrt{a}$, is called a surd, because its value cannot be exactly determined in algebraical symbols free from the radical sign, although the value of a may be such that $\sqrt{a}$ is not really a surd. For instance, when $a=4$, $\sqrt{a}=2$; $\sqrt{a}$ is therefore not really a surd for that particular value of a.

27. Simple and Compound Surds.

Monomial surds or surds consisting of a single term only are called *simple* surds. Thus, $\sqrt{2}$, $5.\sqrt[3]{4}$, $\sqrt[n]{a^2b^4c^3}$, etc., are simple surds.

Expressions which contain two or more simple surds connected by the signs + or −, or contain rational quantities and simple surds so connected, are called *compound* surds. Thus, $2\sqrt{3}+3\sqrt{7}$, $2\sqrt{3}-3\sqrt{7}$, $5+\sqrt{3}$, $2-\sqrt{3}+\sqrt{5}$ are all compound surds.

28. Pure and Mixed Surds.

A simple surd may consist wholly of an irrational quantity or may consist partly of a rational quantity and partly of an irrational quantity as factors. In the former case it is called a *pure* (or *complete*) surd, in the latter case a *mixed* surd.

Thus $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[n]{a^2b^3}$, etc., are pure or complete surds; and $5\sqrt{3}$, $7.\sqrt[3]{5}$, $a\sqrt{b}$, are mixed surds.

29. Surds of different Orders.

Every surd is equivalent to a base raised to a fractional power; and the *order* of a surd is indicated by the root-sign or the denominator of the fractional index.

Thus $\sqrt{2}$, $\sqrt{3}$, $7^{\frac{3}{2}}$, etc., are surds of the second order; $\sqrt[3]{4}$, $5^{\frac{2}{3}}$, $\sqrt[3]{12}$, etc., are surds of the third order; $7^{\frac{3}{4}}$, $\sqrt[4]{27}$, etc., are surds of the fourth order; and $\sqrt[n]{2}$, $\sqrt[n]{12}$, etc, are surds of the nth order.

Surds of the second order are often called *quadratic surds*; surds of the third order, *cubic surds*; and those of the fourth order, *biquadratic surds*.

Surds of the same order are called *equi-radical surds*.

Thus $\sqrt{2}$, $a^{\frac{3}{2}}$, $\sqrt{a+b}$, and $\sqrt{b^3}$ are equi-radical surds; but $a^{\frac{1}{2}}$ and $a^{\frac{2}{3}}$ are not.

30. Rational quantities in Surd form.

It is evident that any rational quantity may be expressed in a surd form of any desired order. The method is illustrated below.

Example 1. *Express 3 as a biquadratic surd.*

Since $3^4=3.3.3.3=81$, we have $3=\sqrt[4]{(3^4)}=\sqrt[4]{81}$.

Example 2. *Express $a+b$ as a surd of the nth order.*

Since $(a+b)$ to the *nth* power$=(a+b)^n$, we have

$$a+b=\sqrt[n]{(a+b)^n}.$$

31. Reduction and Comparison of Surds.

(*i*) *A mixed surd can always be reduced to a pure or complete surd.*

The following example will show the method.

Example 1. *Express* $3\sqrt{7}$, $5.\sqrt[3]{4}$, $a.\sqrt[n]{b}$ as complete surds.

$3\sqrt{7}=\sqrt{3^2}\times\sqrt{7}=\sqrt{3^2\times 7}=\sqrt{9\times 7}=\sqrt{63}.$

$5.\sqrt[3]{4}=\sqrt[3]{5^3}\times\sqrt[3]{4}=\sqrt[3]{5^3\times 4}=\sqrt[3]{125\times 4}=\sqrt[3]{500}.$

$a.\sqrt[n]{b}=\sqrt[n]{a^n}.\sqrt[n]{b}=\sqrt[n]{a^n b}.$

(*ii*) *A pure or complete surd may sometimes be reduced to a mixed surd.*

When this reduction is made, and the quantity is expressed as a mixed surd, *i.e.*, as the product of a rational factor and an irreducible irrational or surd factor, the quantity is said to be *simplified*, and no further simplification is possible.

The method is shown below.

Example 2. *Express as mixed surds :* $\sqrt{45}$; $\sqrt[3]{56}$; $\sqrt[n]{x^{3n}y}$.

$$\sqrt{45}=\sqrt{9\times 5}=(3^2\times 5)^{\frac{1}{2}}=3\times 5^{\frac{1}{2}}=3\sqrt{5}.$$

$$\sqrt[3]{56}=\sqrt[3]{8\times 7}=(2^3\times 7)^{\frac{1}{3}}=2\times 7^{\frac{1}{3}}=2.\sqrt[3]{7}.$$

$$\sqrt[n]{x^{3n}y}=(x^{3n}\times y)^{\frac{1}{n}}=(x^{3n})^{\frac{1}{n}}\times y^{\frac{1}{n}}=x^3\times y^{\frac{1}{n}}=x^3.\sqrt[n]{y}.$$

(*iii*) *Surds of different orders can always be reduced to surds of the same lowest order.*

The process consists in replacing the surds by fractional indices, and then reducing the indices to their lowest common denominator and then going back to the surd form.

Example 3. *Reduce* $\sqrt{2}$ *and* $\sqrt[3]{5}$ *to surds of the same lowest order.*

Here $\sqrt{2}=2^{\frac{1}{2}}$ and $\sqrt[3]{5}=5^{\frac{1}{3}}$.

The L. C. M. of the denominators 2 and 3 is 6.

Hence they can be reduced to surds of the 6th order.

Thus $\sqrt{2}=2^{\frac{1}{2}}=2^{\frac{3}{6}}=\sqrt[6]{2^3}=\sqrt[6]{8}$;

$\sqrt[3]{5}=5^{\frac{1}{3}}=5^{\frac{2}{6}}=\sqrt[6]{5^2}=\sqrt[6]{25}.$

$\therefore$ the reqd. surds are $\sqrt[6]{8}$ and $\sqrt[6]{25}$.

(*iv*). *Comparison of surds of different orders.*

To compare surds of different orders we have to reduce them to the same lowest order.

Example 4. *Which is greater,* $\sqrt{2}$ or $\sqrt[3]{3}$?

$$\sqrt{2}=2^{\frac{1}{2}}=2^{\frac{3}{6}}=\sqrt[6]{2^3}=\sqrt[6]{8}.$$

$$\sqrt[3]{3}=3^{\frac{1}{3}}=3^{\frac{2}{6}}=\sqrt[6]{3^2}=\sqrt[6]{9}.$$

Since 9 is greater than 8, we have $\sqrt[6]{9}$ greater than $\sqrt[6]{8}$, *i.e.*, $\sqrt[3]{3}$ is greater than $\sqrt{2}$.

32. Similar and Dissimilar (or Like and Unlike) Surds. Two or more simple surds are said to be *similar* or *like* when they have the same irrational factor, after reduction, if necessary; while those which have not the same irrational factor common, are said to be *dissimilar* or *unlike* surds.

Thus $\sqrt{3}$, $2\sqrt{3}$, $\sqrt{48}$, $\sqrt{75}$ and $\sqrt{147}$ are similar surds, for the last three, when reduced, become equal to $4\sqrt{3}$, $5\sqrt{3}$ and $7\sqrt{3}$ respectively, and therefore have the same irrational factor $\sqrt{3}$ as the first two; but $\sqrt{20}$ and $\sqrt{72}$ are dissimilar surds, for they, when reduced, become equal to $2\sqrt{5}$ and $6\sqrt{2}$ respectively, and therefore have different irrational factors, namely, $\sqrt{5}$ and $\sqrt{2}$.

EXERCISE 5.

1. Express

(i) 4 as a cubic surd. (ii) 5 as a biquadratic surd.

(iii) $\frac{1}{2}$ as a surd of the 6th order.

(iv) $\sqrt{a+b}$ as a biquadratic surd.

2. Express the following in the form of pure surds :

(i) $4\sqrt{3}$. (ii) $5\sqrt{11}$. (iii) $2.\sqrt[3]{5}$. (iv) $7.\sqrt[4]{3}$.

(v) $a^2.\sqrt[n]{b^3}$. (vi) $x^2.\sqrt[3]{y}$. (vii) $a^3.\sqrt[7]{b^2}$. (viii) $a^2b^3.\sqrt[n]{c}$.

3. Reduce the following to the form of mixed surds :

(i) $\sqrt{72}$. (ii) $\sqrt{98}$. (iii) $\sqrt[3]{-9317}$.

(iv) $\sqrt[3]{a^5b^4}$. (v) $\sqrt[3]{24a^6b^5}$. (vi) $\sqrt[4]{162x^3y^5z^9}$.

4. Reduce to surds of the same lowest order :

(i) $\sqrt{2}$ and $\sqrt[3]{3}$. (ii) $\sqrt[3]{2}$ and $\sqrt[4]{3}$.

(iii) $\sqrt[3]{2}$, $\sqrt[4]{3}$ and $\sqrt[3]{5}$. (iv) $\sqrt{3}$, $\sqrt[3]{4}$ and $\sqrt[6]{2}$.

(v) $\sqrt{a}$, $\sqrt[3]{b^2}$ and $\sqrt[6]{c}$. (vi) $\sqrt[3]{x}$, $\sqrt[5]{y^3}$ and $\sqrt[10]{z^7}$.

5. Which is greater :

(i) $\sqrt{3}$ or $\sqrt[3]{5}$? (ii) $\sqrt{7}$ or $\sqrt[3]{19}$? (iii) $\sqrt[3]{3}$ or $\sqrt[4]{4}$?
(iv) $\sqrt[3]{7}$ or $\sqrt[4]{17}$? (v) $\sqrt[3]{4}$ or $\sqrt[4]{5}$? (vi) $\sqrt[3]{9}$ or $\sqrt[4]{19}$?

6. Arrange in ascending order of magnitude :

(i) $\sqrt[3]{5}$, $\sqrt{2}$ and $\sqrt[4]{7}$. (ii) $\sqrt[3]{7}$, $\sqrt[4]{13}$ and $\sqrt[6]{47}$.
(iii) $\sqrt[3]{2}$, $\sqrt[4]{3}$ and $\sqrt[6]{5}$. (iv) $\sqrt{13}$, $\sqrt[3]{46}$ and $\sqrt[4]{82}$.

7. Show that the following are similar surds :

(i) $\sqrt{12}$, $\sqrt{75}$ and $\sqrt{192}$. (ii) $\sqrt{18}$, $\sqrt{32}$, $\sqrt{98}$ and $\sqrt{72}$.
(iii) $\sqrt[3]{54}$, $\sqrt[3]{250}$ and $\sqrt[3]{432}$. (iv) $\sqrt[3]{24}$, $\sqrt[3]{81}$ and $\sqrt[3]{375}$.
(v) $\sqrt{b^2c^2a}$, $\sqrt{b^4c^6a}$ and $\sqrt{a(b+c)^2}$.

8. Show that the following are dissimilar surds :

(i) $\sqrt{50}$ and $\sqrt{15}$. (ii) $\sqrt{20}$, $\sqrt{18}$ and $\sqrt{32}$.
(iii) $\sqrt{212}$, $\sqrt{147}$ and $\sqrt{180}$. (iv) $\sqrt[3]{56}$, $\sqrt[3]{32}$ and $\sqrt[3]{81}$.

33. Addition and Subtraction of Surds.

Only similar surds can be added or subtracted ; and it is done by collecting the co-efficients (with their proper signs) of the common irrational (or surd) factor.

Dissimilar surds can only be connected by the + and − signs, but cannot be further simplified.

Thus, $2\sqrt{3}+5\sqrt{3}=(2+5)\sqrt{3}=7\sqrt{3}$;

$7\sqrt{2}-3\sqrt{2}=(7-3)\sqrt{2}=4\sqrt{2}$;

$3\sqrt[3]{5}-7\sqrt[3]{5}+\sqrt[3]{5}=(3-7+1).\sqrt[3]{5}=-3\sqrt[3]{5}$; and so on.

Example 1. *Simplify* $\sqrt{108}+\sqrt{75}$.

$$\sqrt{108}+\sqrt{75}=\sqrt{36\times 3}+\sqrt{25\times 3}=6\sqrt{3}+5\sqrt{3}=11\sqrt{3}.$$

Example 2. *Simplify* $\sqrt[3]{24}-\sqrt[3]{192}+\sqrt[3]{1029}$.

The given expression $=\sqrt[3]{8\times 3}-\sqrt[3]{64\times 3}+\sqrt[3]{343\times 3}$

$=2.\sqrt[3]{3}-4.\sqrt[3]{3}+7.\sqrt[3]{3}$

$=(2-4+7).\sqrt[3]{3}$

$=5\sqrt[3]{3}$.

Example 3. *Simplify* $\sqrt{32}-\sqrt{147}+\sqrt{50}+\sqrt{12}$.

The given expression $=4\sqrt{2}-7\sqrt{3}+5\sqrt{2}+2\sqrt{3}$

$=(4+5)\sqrt{2}-(7-2)\sqrt{3}$

$=9\sqrt{2}-5\sqrt{3}$.

EXERCISE 6.

Simplify :

1. $\sqrt{50}+\sqrt{98}$.
2. $\sqrt{45}-\sqrt{20}$.
3. $\sqrt[3]{40}+\sqrt[3]{135}$.
4. $\sqrt[3]{108}-\sqrt[3]{32}$.
5. $\sqrt{72}-\sqrt{50}+\sqrt{18}$.
6. $\sqrt[3]{56}-\sqrt[3]{875}+\sqrt[3]{189}$.
7. $\sqrt{a^3b}+\sqrt{ab^3}+\sqrt{abc^2}$.
8. $x.\sqrt[3]{y^3z^3a}+y.\sqrt[3]{-27z^3x^3a}-z.\sqrt[3]{-64x^3y^3a}$.

34. Multiplication of Simple Surds.

To multiply two or more simple surds of the same order, multiply the rational and irrational factors separately. The irrational factors are multiplied together by the formula

$$\sqrt[n]{a}.\sqrt[n]{b}=a^{\frac{1}{n}}.b^{\frac{1}{n}}=(ab)^{\frac{1}{n}}=\sqrt[n]{ab}.$$

To multiply simple surds of different orders, transform them into equivalent surds of the same lowest order, and then proceed as before.

Example 1. *Multiply* $3\sqrt{2}$ *by* $5\sqrt{3}$.

Here, the given surds are of the same order.

$\therefore\quad 3\sqrt{2}\times 5\sqrt{3}=3\times 5\times\sqrt{2}\times\sqrt{3}=15\sqrt{2\times 3}=15\sqrt{6}$.

Example 2. *Multiply* $4.\sqrt[3]{3}$ by $5\sqrt[4]{5}$.

Here, the surds are of different orders.

$$\therefore\quad 4.\sqrt[3]{3}\times 5.\sqrt[4]{5}=4\times 5\times 3^{\frac{1}{3}}\times 5^{\frac{1}{4}}=20\times 3^{\frac{4}{12}}\times 5^{\frac{3}{12}}$$
$$=20\times(3^4\times 5^3)^{\frac{1}{12}}=20\times(81\times 125)^{\frac{1}{12}}$$
$$=20\times(10125)^{\frac{1}{12}}=20.\sqrt[12]{10125}.$$

EXERCISE 7.

Simplify :

1. $\sqrt{14}\times\sqrt{21}$.
2. $2\sqrt{3}\times 3\sqrt{5}$.
3. $\sqrt{20}\times\sqrt{45}$.
4. $4\sqrt{3}\times\sqrt{8}$.
5. $\sqrt{2ax}\times\sqrt{3bx}$.
6. $2.\sqrt[3]{x}\times 5.\sqrt[3]{3x^2}$.
7. $2.\sqrt[3]{4}\times 3\sqrt{8}$.
8. $\sqrt[4]{27}\times\sqrt[3]{81}$.
9. $5.\sqrt[3]{2}\times 2\sqrt{5}$.
10. $\sqrt[3]{12}\times\sqrt{27}$.
11. $\sqrt[3]{54}\times\sqrt[8]{128}$.
12. $5.\sqrt[3]{6}\times 3\sqrt{10}$.

35. Multiplication of Compound Surds.

In finding the product of two or more compound surds, we follow the same method as in dealing with rational polynomials.

Example 1. *Multiply* $2\sqrt{5}+3\sqrt{2}$ by $7\sqrt{5}-5\sqrt{2}$.

$(2\sqrt{5}+3\sqrt{2})(7\sqrt{5}-5\sqrt{2})$

$$=(2\sqrt{5})(7\sqrt{5})+(3\sqrt{2})(7\sqrt{5})-(2\sqrt{5})(5\sqrt{2})-(3\sqrt{2})(5\sqrt{2})$$
$$=70+21\sqrt{10}-10\sqrt{10}-30$$
$$=40+11\sqrt{10}.$$

Example 2. *Multiply* $3\sqrt{5}+2\sqrt{11}$ by $3\sqrt{5}-2\sqrt{11}$.

$$(3\sqrt{5}+2\sqrt{11})(3\sqrt{5}-2\sqrt{11})=(3\sqrt{5})^2-(2\sqrt{11})^2$$
$$=9.5-4.11.$$
$$=45-44=1.$$

Example 3. *Find the square of* $\sqrt{a+b}+\sqrt{a-b}$.

$$(\sqrt{a+b}+\sqrt{a-b})^2=(\sqrt{a+b})^2+(\sqrt{a-b})^2+2\sqrt{a+b}.\sqrt{a-b}$$
$$=a+b+a-b+2\sqrt{(a+b)(a-b)}$$
$$=2a+2\sqrt{a^2-b^2}.$$

Example 4. *Multiply* $1+\sqrt{2}-\sqrt{3}$ by $1-\sqrt{2}+\sqrt{3}$.

$$(1+\sqrt{2}-\sqrt{3})(1-\sqrt{2}+\sqrt{3})=1^2-(\sqrt{2}-\sqrt{3})^2$$
$$=1-(2+3-2\sqrt{6})$$
$$=1-5+2\sqrt{6}$$
$$=2\sqrt{6}-4.$$

EXERCISE 8

Simplify :

1. $(\sqrt{2}+\sqrt{3})\times\sqrt{6}$.
2. $(\sqrt{5}-\sqrt{3})\sqrt{15}$.
3. $(3\sqrt{2}+4\sqrt{3})(2\sqrt{2}+5\sqrt{3})$.
4. $(2\sqrt{3}-3\sqrt{5})(3\sqrt{3}+4\sqrt{5})$.
5. $(3\sqrt{2}+2\sqrt{3})(\sqrt{6}-\sqrt{3})$.
6. $(2\sqrt{15}-\sqrt{6})(\sqrt{5}+2\sqrt{2})$.
7. $(2\sqrt{7}+\sqrt{5})(3\sqrt{5}-\sqrt{7})$.
8. $(3-2\sqrt{2})(7+5\sqrt{2})$.
9. $(7\sqrt{2}+3\sqrt{10})(7\sqrt{2}-3\sqrt{10})$.
10. $(2\sqrt{19}+5\sqrt{3})(2\sqrt{19}-5\sqrt{3})$.
11. $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$.
12. $(2\sqrt{a}+5\sqrt{b})(3\sqrt{a}-4\sqrt{b})$.
13. $(x+\sqrt{x^2-1})(x-\sqrt{x^2-1})$.

14. $(\sqrt{a+x}+\sqrt{a})(\sqrt{a+x}-\sqrt{a})$.

15. $(1+\sqrt{2}-\sqrt{3})(2+\sqrt{2}+\sqrt{6})$.

16. $(\sqrt{2}+\sqrt{3}+\sqrt{5})(\sqrt{2}-\sqrt{3}+\sqrt{5})$.

17. $(\sqrt{a}+\sqrt{a-b}+\sqrt{b})(\sqrt{a}-\sqrt{a-b}+\sqrt{b})$.

18. $(\sqrt{3}+\sqrt{6}-\sqrt{2})(\sqrt{6}+\sqrt{2}-\sqrt{3})(\sqrt{2}+\sqrt{3}-\sqrt{6})$.

19. $(\sqrt{6}-1)(\sqrt{3}+2)(2\sqrt{2}-\sqrt{3})$. [Cal. F. A. 1883]

Find the square of :

20. $2-\sqrt{3}$. **21.** $\sqrt{2}+\sqrt{5}$. **22.** $5\sqrt{7}-4\sqrt{5}$.

23. $\sqrt{x+1}-\sqrt{x-1}$. **24.** $\sqrt{3x-5}-\sqrt{2x+1}$.

36. Rationalization of Surds.

It is sometimes found necessary in the course of algebraic operations to render a surd rational by multiplying it by some suitable expression. This process is called *rationalization*; and the expression or factor which is required to rationalize a given surd is called its *rationalizing factor.* It is evident that the rationalizing factor of a surd must itself be a surd.

For example :

$$2\sqrt{3}\times\sqrt{3}=6\ ;$$
$$(2+\sqrt{3})(2-\sqrt{3})=4-3=1\ ;$$
$$(4\sqrt{7}+3\sqrt{5})(4\sqrt{7}-3\sqrt{5})=112-45=67\ ;\ \text{etc.}$$

Thus, $2\sqrt{3}$ is rationalized by $\sqrt{3}$, $2+\sqrt{3}$ by $2-\sqrt{3}$, and $4\sqrt{7}+3\sqrt{5}$ by $4\sqrt{7}-3\sqrt{5}$. Here $\sqrt{3}$, $2-\sqrt{3}$, $4\sqrt{7}-3\sqrt{5}$ are the rationalizing factors.

37. Conjugate and Complementary Surds.

The most frequent and important case of rationalization that occurs usually in practice is in connection with *binomial quadratic surds*.

It is evident that if $\sqrt{a}+\sqrt{b}$ is such a surd it can be rationalized by the factor $\sqrt{a}-\sqrt{b}$; for

$$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b$$

Similarly, $a+\sqrt{b}$ is rationalized by $a-\sqrt{b}$ and $3\sqrt{5}+2\sqrt{3}$ by $3\sqrt{5}-2\sqrt{3}$; and so on.

Two binomial quadratic surds, like $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$, which are such that they differ only in the sign connecting the two terms, are said to be *conjugate* or *complementary* to each other.

It follows from what has been proved above that *the product of any pair of conjugate surds is always rational; i.e., the rationalizing factor of any binomial quadratic surd is its conjugate.*

Example 1. *Express* $\frac{1}{2\sqrt{5}-3\sqrt{2}}$ *with a rational denominator.*

$$\frac{1}{2\sqrt{5}-3\sqrt{2}}=\frac{2\sqrt{5}+3\sqrt{2}}{(2\sqrt{5}-3\sqrt{2})(2\sqrt{5}+3\sqrt{2})}$$
$$=\frac{2\sqrt{5}+3\sqrt{2}}{(2\sqrt{5})^2-(3\sqrt{2})^2}$$
$$=\frac{2\sqrt{5}+3\sqrt{2}}{20-18}$$
$$=\tfrac{1}{2}(2\sqrt{5}+3\sqrt{2}).$$

Example 2. *Rationalize the denominator of* $\frac{1}{1+\sqrt{2}+\sqrt{3}}$.

$$\frac{1}{1+\sqrt{2}+\sqrt{3}}=\frac{1}{1+\sqrt{2}+\sqrt{3}}\times\frac{1+\sqrt{2}-\sqrt{3}}{1+\sqrt{2}-\sqrt{3}}$$
$$=\frac{1+\sqrt{2}-\sqrt{3}}{(1+\sqrt{2})^2-(\sqrt{3})^2}$$
$$=\frac{1+\sqrt{2}-\sqrt{3}}{1+2\sqrt{2}+2-3}$$
$$=\frac{1+\sqrt{2}-\sqrt{3}}{2\sqrt{2}}$$
$$=\frac{1+\sqrt{2}-\sqrt{3}}{2\sqrt{2}}\times\frac{\sqrt{2}}{\sqrt{2}}$$
$$=\frac{\sqrt{2}+2-\sqrt{6}}{4}.$$

Example 3. *Rationalize the denominator of* $\frac{\sqrt{x+a}-\sqrt{x-a}}{\sqrt{x+a}+\sqrt{x-a}}$.

The given expression $=\frac{(\sqrt{x+a}-\sqrt{x-a})^2}{(\sqrt{x+a}+\sqrt{x-a})(\sqrt{x+a}-\sqrt{x-a})}$

$$=\frac{x+a+x-a-2\sqrt{(x+a)(x-a)}}{(x+a)-(x-a)}$$
$$=\frac{2x-2\sqrt{x^2-a^2}}{2a}$$
$$=\frac{x-\sqrt{x^2-a^2}}{a}.$$

38. Division of Surds.

In division by a simple surd, proceed on the same principle as in multiplication by a simple surd [Art. 34]; and if after simplification there appears a surd in the divisor or denominator, rationalize it.

In division by a compound surd, the divisor has to be rationalized.

Example 1. *Divide* $3\sqrt{2}$ *by* $\sqrt[3]{3}$.

$$3\sqrt{2}\div\sqrt[3]{3}=\frac{3\times2^{\frac{1}{2}}}{3^{\frac{1}{3}}}=\frac{3\times2^{\frac{1}{2}}\times3^{\frac{2}{3}}}{3^{\frac{1}{3}}\times3^{\frac{2}{3}}}=\frac{3\times2^{\frac{1}{2}}\times3^{\frac{2}{3}}}{3}=2^{\frac{1}{2}}\times3^{\frac{2}{3}}=2^{\frac{3}{6}}\times3^{\frac{4}{6}}$$

$$=\sqrt[6]{2^3\times3^4}=\sqrt[6]{8\times81}=\sqrt[6]{648}.$$

Example 2. *Find the value of* $\sqrt{125}\div\sqrt{15}$ *to 3 places of decimals, given* $\sqrt{3}=1{\cdot}7321$.

$$\sqrt{125}\div\sqrt{15}=\sqrt{\frac{125}{15}}=\sqrt{\frac{25}{3}}=\frac{5}{\sqrt{3}}=\frac{5\times\sqrt{3}}{\sqrt{3}\times\sqrt{3}}=\frac{5\sqrt{3}}{3}=\frac{5\times1{\cdot}7321}{3}$$

$$=\frac{8{\cdot}6605}{3}=2{\cdot}887, \text{ nearly}.$$

Example 3. *Divide* $\sqrt{7}+2\sqrt{3}$ *by* $\sqrt{5}-\sqrt{3}$.

$$\frac{\sqrt{7}+2\sqrt{3}}{\sqrt{5}-\sqrt{3}}=\frac{\sqrt{7}+2\sqrt{3}}{\sqrt{5}-\sqrt{3}}\cdot\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}+\sqrt{3}}$$

$$=\frac{\sqrt{35}+\sqrt{21}+2\sqrt{15}+6}{5-3}$$

$$=\tfrac{1}{2}(6+2\sqrt{15}+\sqrt{21}+\sqrt{35}).$$

EXERCISE 9.

1. Divide :

(i) $\sqrt[4]{12}$ by $\sqrt{27}$. (ii) $2.\sqrt[3]{4}$ by $3\sqrt{8}$.

(iii) $5\sqrt[3]{6}$ by $3\sqrt{10}$. (iv) $5.\sqrt[3]{2}$ by $2\sqrt{5}$.

(v) $4.\sqrt[3]{3}$ by $3\sqrt{5}$. (vi) $\sqrt[4]{27}$ by $\sqrt[3]{81}$.

2. Divide :

(i) $2+\sqrt{3}$ by $3+\sqrt{3}$. (ii) $\sqrt{5}+\sqrt{3}$ by $4+\sqrt{15}$.

(iii) $7\sqrt{3}-5\sqrt{2}$ by $4\sqrt{3}+3\sqrt{2}$. (iv) $8-5\sqrt{2}$ by $4+\sqrt{2}$.

Also find the quotients to three places of decimals, being given that $\sqrt{2}=1{\cdot}4142$, $\sqrt{3}=1{\cdot}7321$, $\sqrt{5}=2{\cdot}2361$ and $\sqrt{6}=2{\cdot}4495$.

3. Divide $2x^3-6x+5$ by $\sqrt[3]{2}x+\sqrt[3]{4}+1$. [Cal. F. A. 1885]

4. Express the following with a rational denominator :

(i) $\dfrac{1}{5\sqrt{2}-4\sqrt{3}}$. (ii) $\dfrac{1}{3\sqrt{5}+2\sqrt{11}}$.

(iii) $\dfrac{3\sqrt{2}+2\sqrt{3}}{3\sqrt{2}-2\sqrt{3}}$. (iv) $\dfrac{1}{1+\sqrt{2}-\sqrt{3}}$.

(v) $\dfrac{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}$. (vi) $\dfrac{x+1+\sqrt{x^2+1}}{x+1-\sqrt{x^2+1}}$.

(vii) $\dfrac{x+y+\sqrt{(x^2-y^2)}}{x+y-\sqrt{(x^2-y^2)}}$. (viii) $\dfrac{1}{\sqrt{a}+\sqrt{b}+\sqrt{c}}$.

5. Simplify :

(i) $\dfrac{\sqrt{3}-1}{\sqrt{3}+1}+\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$. (ii) $\dfrac{3+\sqrt{5}}{3-\sqrt{5}}+\dfrac{3-\sqrt{5}}{3+\sqrt{5}}$

(iii) $\dfrac{1}{x+\sqrt{(x^2-1)}}+\dfrac{1}{x-\sqrt{(x^2-1)}}$.

(iv) $\dfrac{1}{a-\sqrt{(a^2-x^2)}}-\dfrac{1}{a+\sqrt{(a^2-x^2)}}$.

(v) $\dfrac{1-\sqrt{2}+\sqrt{3}}{1+\sqrt{2}+\sqrt{3}}-\dfrac{1-\sqrt{2}-\sqrt{3}}{1+\sqrt{2}-\sqrt{3}}$.

(vi) $\dfrac{\sqrt{(x^2+1)}+\sqrt{(x^2-1)}}{\sqrt{(x^2+1)}-\sqrt{(x^2-1)}}+\dfrac{\sqrt{(x^2+1)}-\sqrt{(x^2-1)}}{\sqrt{(x^2+1)}+\sqrt{(x^2-1)}}$.

6. Calculate $\dfrac{\sqrt{2(2+\sqrt{3})}}{\sqrt{3}(\sqrt{3}+1)}-\dfrac{\sqrt{2(2-\sqrt{3})}}{\sqrt{3}(\sqrt{3}-1)}$, correct to 2 places of decimals. [Dac. 1933]

7. Find the value of $\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$, when $x=\dfrac{\sqrt{3}}{2}$.

[Bom. P. E. 1883]

39. Theorems concerning Quadratic Surds.

(*i*) *A simple quadratic surd cannot be equal to the sum or difference of a rational quantity and a simple quadratic surd.*

If possible, let $\sqrt{x}=a\pm\sqrt{b}$ where a is rational and $\sqrt{x}$, $\sqrt{b}$ are quadratic surds.

Then squaring both sides, we must have

$$x=a^2+b\pm 2a\sqrt{b}$$

$$\therefore \qquad \sqrt{b}=\pm\frac{x-a^2-b}{2a}.$$

Thus a surd becomes equal to a rational quantity, which is impossible.

(ii) *A simple quadratic surd cannot be equal to the sum or difference of two simple quadratic surds.*

If possible, let $\sqrt{x}=\sqrt{a}\pm\sqrt{b}$, where $\sqrt{x}$, $\sqrt{a}$, $\sqrt{b}$ are all simple quadratic surds.

Squaring, we have $x=a+b\pm 2\sqrt{ab}$

$$\therefore \qquad \sqrt{ab}=\pm\frac{x-a-b}{2}.$$

Thus a surd becomes equal to a rational quantity, which is impossible.

(iii) *The product of two dissimilar quadratic surds cannot be rational.*

If possible, let $\sqrt{a}.\sqrt{b}=p$ where p is a rational quantity.

Then $$\sqrt{a}=\frac{p}{\sqrt{b}}=\frac{p}{b}\sqrt{b},$$

so that $\sqrt{a}$ and $\sqrt{b}$ are not dissimilar.

Hence, the product of two dissimilar quadratic surds cannot be rational.

This property is evidently true of any number of surds and of any order.

(iv) *If $a+\sqrt{b}=x+\sqrt{y}$, where a and x are both rational and $\sqrt{b}$ and $\sqrt{y}$ are both irrational, then $a=x$ and $b=y$.*

For, if a is not equal to x, let $a=x+m$; then

$$x+m+\sqrt{b}=x+\sqrt{y}.$$

$$\therefore \qquad m+\sqrt{b}=\sqrt{y}$$

that is, the surd $\sqrt{y}$ is equal to the sum of a rational quantity m and an irrational quantity $\sqrt{b}$, which is impossible, by Theorem (i).

Hence $a=x$, and so $\sqrt{b}=\sqrt{y}$, or $b=y$.

Cor. Since, if $a+\sqrt{b}=x+\sqrt{y}$, we have $a=x$ and $b=y$, it follows that $a-\sqrt{b}=x-\sqrt{y}$.

It is thus evident that an equation like $x+\sqrt{y}=a+\sqrt{b}$ or. $x-\sqrt{y}=a-\sqrt{b}$, which contains both rational and irrational quantities, is really equivalent to two equations, namely $x=a$ and $y=b$, which are obtained by equating the rational and the irrational parts separately.

(v) *If* $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$, *then* $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.
For, squaring both sides, we have

$$a+\sqrt{b}=x+y+2\sqrt{xy}.$$

$$\therefore \quad \left.\begin{array}{l} a=x+y \\ \text{and} \quad \sqrt{b}=2\sqrt{xy} \end{array}\right\} \qquad \text{[Theorem (iv)]}$$

Hence by subtraction,

$$a-\sqrt{b}=x+y-2\sqrt{xy}.$$

$$\therefore \quad \sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}.$$

40. Square Roots of Quadratic Surds.

(i) *To find the square root of a binomial quadratic surd.*

The square root of a binomial quadratic surd may *sometimes* be obtained in the form of a binomial quadratic surd. The method is shown below.

Let the binomial quadratic surd be of the form $a+\sqrt{b}$, where a is rational and $\sqrt{b}$ a surd.

Let us suppose, if possible, that it has got a square root of the form $\sqrt{x}+\sqrt{y}$, where x and y are rational and $x>y$.

Hence, we have

$$a+\sqrt{b}=(\sqrt{x}+\sqrt{y})^2=x+y+2\sqrt{xy}$$

Hence, equating the rational and irrational parts separately, we have

$$\left.\begin{array}{ll} a=x+y \cdots & (1) \\ \text{and} \quad \sqrt{b}=2\sqrt{xy}\cdots & (2) \end{array}\right\} \qquad \text{[Theorem (iv)]}$$

Squaring (1) and (2) and subtracting the latter from the former we have

$$a^2-b=(x+y)^2-4xy=(x-y)^2$$

$$\therefore \quad \sqrt{a^2-b}=x-y.$$

$$\text{Thus,} \quad \left.\begin{array}{l} x+y=a \\ \text{and} \quad x-y=\sqrt{a^2-b} \end{array}\right\}$$

$$\therefore \quad x=\tfrac{1}{2}(a+\sqrt{a^2-b}), \text{ and } y=\tfrac{1}{2}(a-\sqrt{a^2-b}).$$

$$\therefore \quad \sqrt{(a+\sqrt{b})}=\sqrt{x}+\sqrt{y}$$

$$=\sqrt{\{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{a^2-b}\}}+\sqrt{\{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{a^2-b}\}}.$$

Similarly,

$$\sqrt{(a-\sqrt{b})}=\sqrt{x}-\sqrt{y}$$
$$=\sqrt{\{\tfrac{1}{2}a+\tfrac{1}{2}\sqrt{a^2-b}\}}-\sqrt{\{\tfrac{1}{2}a-\tfrac{1}{2}\sqrt{a^2-b}\}}.$$

It follows at once that only when a^2-b is a perfect square do we get rational values of x and y, and so the square root of $a+\sqrt{b}$ remains a quadratic surd. If however a^2-b be not a perfect square, then the square root is not a quadratic surd at all, and the form obtained is of very little practical use.

If the given binomial quadratic surd is of the form $\sqrt{a}+\sqrt{b}$ where $\sqrt{a}$ and $\sqrt{b}$ are both surds, then the method of finding out the square root is similar. But the square root is no longer a quadratic surd.

Method of Inspection. If the given expression is of the form $a+2\sqrt{b}$, then we can find out the square root $\sqrt{x}+\sqrt{y}$ in many cases by inspection, for, we have simply to find out x and y in such a manner that $x+y=a$, and $xy=b$.

(*ii*) *To find the square root of a quadratic surd of the form* $a+\sqrt{b}+\sqrt{c}+\sqrt{d}$.

The square root of a compound quadratic surd of the form $a+\sqrt{b}+\sqrt{c}+\sqrt{d}$ can *sometimes* be found in the form $\sqrt{x}+\sqrt{y}+\sqrt{z}$, where x, y, z are rational. The method is shown below.

Let $a+\sqrt{b}+\sqrt{c}+\sqrt{d}=(\sqrt{x}+\sqrt{y}+\sqrt{z})^2$

$$=x+y+z+2\sqrt{yz}+2\sqrt{zx}+2\sqrt{xy}.$$

This equality will be satisfied, if we can find out x, y, z so that

$$x+y+z=a \quad \cdots \quad \cdots \quad (1)$$
$$2\sqrt{yz}=\sqrt{b} \quad \cdots \quad \cdots \quad (2)$$
$$2\sqrt{zx}=\sqrt{c} \quad \cdots \quad \cdots \quad (3)$$
$$2\sqrt{xy}=\sqrt{d} \quad \cdots \quad \cdots \quad (4)$$

From (2) and (3), by multiplication

$$4z\sqrt{xy}=\sqrt{bc}.$$

Hence from (4), $4z.\dfrac{\sqrt{d}}{2}=\sqrt{bc}$ $\therefore$ $z=\dfrac{1}{2}\sqrt{\dfrac{bc}{d}}$.

Similarly, $x=\dfrac{1}{2}\sqrt{\dfrac{cd}{b}}$, $y=\dfrac{1}{2}\sqrt{\dfrac{db}{c}}$.

Hence, the required square root

$$=\sqrt{\left(\frac{1}{2}\sqrt{\frac{cd}{b}}\right)}+\sqrt{\left(\frac{1}{2}\sqrt{\frac{db}{c}}\right)}+\sqrt{\left(\frac{1}{2}\sqrt{\frac{bc}{d}}\right)} \quad \cdots \quad (5)$$

But in order that this may be the square root, the values of x, y, z obtained must satisfy equation (1).

$$\frac{1}{2}\sqrt{\frac{cd}{b}}+\frac{1}{2}\sqrt{\frac{db}{c}}+\frac{1}{2}\sqrt{\frac{bc}{d}}=a\ ;$$

$$bc+cd+db=2a\sqrt{bcd} \qquad \cdots \qquad \cdots \qquad (6)$$

which is therefore the condition that the square root of the given expression can be found at all.

Further it is evident from (5) that in order that x, y, z may be rational, $\frac{cd}{b}$, $\frac{db}{c}$, $\frac{bc}{d}$ must all be perfect squares.

Example 1. *Find the square root of* $21+8\sqrt{5}$.

Let $\qquad \sqrt{21+8\sqrt{5}}=\sqrt{x}+\sqrt{y}$

Squaring both sides, $21+8\sqrt{5}=x+y+2\sqrt{xy}$.

Hence equating the rational and irrational parts separately, we have

$$x+y=21 \qquad \cdots \qquad \cdots \qquad (1)$$

and $\qquad 2\sqrt{xy}=8\sqrt{5}$

or $\qquad xy=80. \qquad \cdots \qquad \cdots \qquad (2)$

Now from equations (1) and (2) it can be easily found by inspection that $x=16$ and $y=5$.

Thus the required square root is $\sqrt{16}+\sqrt{5}=4+\sqrt{5}$.

Example 2. *Find the square root of* $8-2\sqrt{15}$.

Let $\qquad \sqrt{8-2\sqrt{15}}=\sqrt{x}-\sqrt{y}$.

$\therefore \qquad 8-2\sqrt{15}=x+y-2\sqrt{xy}$.

$$\therefore \qquad \left.\begin{aligned} x+y&=8 \\ xy&=15 \end{aligned}\right\}$$

These two equations are satisfied when $x=5$, and $y=3$.

Thus the required square root $=\sqrt{5}-\sqrt{3}$.

Example 3. *Find the square root of* $10+2\sqrt{6}-2\sqrt{10}-2\sqrt{15}$.

Let the square root be $\sqrt{x}+\sqrt{y}-\sqrt{z}$. Then we have

$$10+2\sqrt{6}-2\sqrt{10}-2\sqrt{15}=(\sqrt{x}+\sqrt{y}-\sqrt{z})^2$$

$$=x+y+z+2\sqrt{xy}-2\sqrt{yz}-2\sqrt{zx}.$$

$$\therefore \qquad x+y+z=10 \quad \ldots \quad \ldots \quad (1)$$
$$2\sqrt{xy}=2\sqrt{6} \quad \ldots \quad \ldots \quad (2)$$
$$2\sqrt{yz}=2\sqrt{10} \quad \ldots \quad \ldots \quad (3)$$
$$2\sqrt{zx}=2\sqrt{15} \quad \ldots \quad \ldots \quad (4)$$

Multiplying (2) and (3), $4y\sqrt{zx}=4\sqrt{60}$.

Dividing by (4), $2y=\dfrac{4\sqrt{60}}{2\sqrt{15}}=2\sqrt{4}=2.2=4. \quad \therefore \quad y=2.$

Hence from (2) and (3), $x=3$, $z=5$

Further, equation (1) is satisfied by these values of x, y, and z. Hence, the required square root $=\sqrt{3}+\sqrt{2}-\sqrt{5}$.

EXERCISE 10.

Find the square root of :

1. $7+4\sqrt{3}$.
2. $11-6\sqrt{2}$. [Cal. F. A. 1884]
3. $8+2\sqrt{15}$.
4. $9-4\sqrt{5}$. [Cal. F. A. 1883]
5. $6+\sqrt{11}$.
6. $28-6\sqrt{3}$. [Cal. F. A. 1861]
7. $\sqrt{15}+\sqrt{27}$.
8. $12-6\sqrt{3}$. [All. 1889]
9. $2\frac{1}{4}+\sqrt{5}$.
10. $4\frac{1}{5}+\dfrac{4}{\sqrt{5}}$.
11. $a+\sqrt{a^2-b^2}$.
12. $x^2+2a\sqrt{(x^2-a^2)}$.
13. $x^2+2(x-1)\sqrt{(2x-1)}$.
14. $ab+2b\sqrt{ab-b^2}$.
15. $x^3+a^3+2ax\sqrt{ax}$.
16. $3(x+1)+2\sqrt{(2x^2+5x+2)}$.
17. $x+y+z+2\sqrt{xy+zx}$.
18. $a+\sqrt{a^2-b^2-c^2+2bc}$.
19. $6+2\sqrt{2}+2\sqrt{3}+2\sqrt{6}$.
20. $8+2\sqrt{2}-2\sqrt{5}-2\sqrt{10}$.

41. We proceed to show below various types of examples involving surds.

Example 1. *If* $x=\dfrac{\sqrt{3}+1}{\sqrt{3}-1}$, $y=\dfrac{\sqrt{3}-1}{\sqrt{3}+1}$, *find the value of*
$$\frac{x^2+xy+y^2}{x^2-xy+y^2}.$$

We have $x^2+y^2=\left(\dfrac{\sqrt{3}+1}{\sqrt{3}-1}\right)^2+\left(\dfrac{\sqrt{3}-1}{\sqrt{3}+1}\right)^2$
$$=\frac{4+2\sqrt{3}}{4-2\sqrt{3}}+\frac{4-2\sqrt{3}}{4+2\sqrt{3}}$$

$$=\frac{2+\sqrt{3}}{2-\sqrt{3}}+\frac{2-\sqrt{3}}{2+\sqrt{3}}$$
$$=\frac{(2+\sqrt{3})^2+(2-\sqrt{3})^2}{4-3}$$
$$=\frac{(7+4\sqrt{3})+(7-4\sqrt{3})}{1}=14.$$

Also $$xy=\frac{\sqrt{3}+1}{\sqrt{3}-1}\times\frac{\sqrt{3}-1}{\sqrt{3}+1}=1.$$

Hence the reqd. value $=\dfrac{14+1}{14-1}=\dfrac{15}{13}$

Example 2. *Find the value of* x^3+y^3, *when*
$$x=\tfrac{1}{2}\left(a+\sqrt{\frac{4b^3-a^3}{3a}}\right) \text{ and } y=\tfrac{1}{2}\left(a-\sqrt{\frac{4b^3-a^3}{3a}}\right).$$

We have $$x+y=\tfrac{1}{2}\left(a+\sqrt{\frac{4b^3-a^3}{3a}}\right)+\tfrac{1}{2}\left(a-\sqrt{\frac{4b^3-a^3}{3a}}\right)=a.$$
$$xy=\tfrac{1}{2}\left(a+\sqrt{\frac{4b^3-a^3}{3a}}\right)\times\tfrac{1}{2}\left(a-\sqrt{\frac{4b^3-a^3}{3a}}\right)$$
$$=\tfrac{1}{4}\left(a^2-\frac{4b^3-a^3}{3a}\right)=\frac{a^3-b^3}{3a}.$$

$$\therefore\quad x^3+y^3=(x+y)^3-3xy(x+y)=a^3-3\cdot\left(\frac{a^3-b^3}{3a}\right)a.$$
$$=a^3-(a^3-b^3)=b^3.$$

Example 3. *If* $(x+\sqrt{x^2-bc})(y+\sqrt{y^2-ca})(z+\sqrt{z^2-ab})$
$$=(x-\sqrt{x^2-bc})(y-\sqrt{y^2-ca})(z-\sqrt{z^2-ab})$$
show that each of the expressions $=\pm abc$ [*Bom. P. E.* 1889]

Let each of the given expressions be equal to k.

Then, multiplying them together, we have
$$k^2=(x+\sqrt{x^2-bc})(y+\sqrt{y^2-ca})(z+\sqrt{z^2-ab})$$
$$\times(x-\sqrt{x^2-bc})(y-\sqrt{y^2-ca})(z-\sqrt{z^2-ab})$$
$$=\{x^2-(x^2-bc)\}\{y^2-(y^2-ca)\}\{z^2-(z^2-ab)\}$$
$$=bc.ca.ab$$
$$=a^2b^2c^2.$$
$$\therefore\quad k=\pm abc.$$

Hence each of the given expressions $=\pm abc$.

Example 4. *Find the real value of* $\sqrt[3]{(\sqrt{5}+2)}-\sqrt[3]{(\sqrt{5}-2)}$.

Let $\sqrt[3]{(\sqrt{5}+2)}-\sqrt[3]{(\sqrt{5}-2)}=x$.

Cubing both sides, we have

$(\sqrt{5}+2)-(\sqrt{5}-2)-3\sqrt[3]{(5-4)}\{\sqrt[3]{(\sqrt{5}+2)}-\sqrt[3]{(\sqrt{5}-2)}\}=x^3$

or, $4-3x=x^3$

or, $x^3+3x-4=0$

or, $(x-1)(x^2+x+4)=0.$

Evidently $x=1$ satisfies this equation, and no other real values of x does.

Hence the real value of the given expression $=1$.

Example 5. *If* $x=3+\sqrt{2}$, *find the value of*

$$x^4-12x^3+55x^2-114x+84.$$

We have $x=3+\sqrt{2}$ $\therefore$ $x-3=\sqrt{2}$.

Raising to the 4th power, we have

$$(x-3)^4=x^4-12x^3+54x^2-108x+81=4.$$

$$\begin{aligned}\therefore\quad & x^4-12x^3+55x^2-114x+84\\ &=(x^4-12x^3+54x^2-108x+81)+(x^2-6x+9)-6\\ &=(x-3)^4+(x-3)^2-6\\ &=4+2-6=0.\end{aligned}$$

Example 6. *Determine the value of*

$$\sqrt{a\sqrt[3]{b\sqrt{a\sqrt[3]{b\cdots}}}}\ \textit{continued to infinity.}$$

[*Bom. P. E.* 1884]

Let $x=\sqrt{a\sqrt[3]{b\sqrt{a\sqrt[3]{b\cdots\cdots}}}}$ to infinity$\cdots$ (1)

Squaring, $x^2=a\sqrt[3]{b\sqrt{a\sqrt[3]{b\cdots\cdots}}}$ to infinity$\cdots$ (2)

Cubing (2), $x^6=a^3b.\sqrt{a\sqrt[3]{b\cdots\cdots\cdots\cdots}}$ to infinity

$=a^3bx.$ ... ... [from (1)]

Removing the common factor x, we have $x^5=a^3b.$

$\therefore$ $x=\sqrt[5]{a^3b}.$

EXERCISE 11.

1. Express $\dfrac{(\sqrt{3}+\sqrt{5})(\sqrt{5}+\sqrt{2})}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$ with a rational denominator.

2. Express $\dfrac{1}{\sqrt{a}+\sqrt{b}+\sqrt{a+b}}$ with a rational denominator.

3. Bring $\dfrac{7}{2^{\frac{1}{2}}+2^{\frac{1}{4}}+1}$ to a form with a rational denominator.

[Bom. P. E. 1890]

4. Simplify $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{(2+\sqrt{3})}}+\dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{(2-\sqrt{3})}}$.

[Bom. P. E. 1888]

5. Show that $\dfrac{1}{\sqrt{(11-\sqrt{30})}}-\dfrac{3}{\sqrt{(7-2\sqrt{10})}}-\dfrac{4}{\sqrt{(8+4\sqrt{3})}}=0.$

6. Find the value of $\dfrac{\sqrt{x^2-1}}{x-\sqrt{x^2-1}}$, when $2x=\sqrt{a}+\dfrac{1}{\sqrt{a}}$.

7. Find the value of x^3+y^3, when $x=\dfrac{2-\sqrt{3}}{2+\sqrt{3}}$ and $y=\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$.

8. Find the value of $\dfrac{1}{2}\cdot\dfrac{(x-1)^2}{x+1}+\dfrac{7}{3}\cdot\dfrac{x+1}{x+2}$, when $x=\sqrt{3}-1$.

[Cal. F. A. 1884]

9. Reduce $\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}$, when $x=\dfrac{\sqrt{3}}{2}$.

[Bom. P. E. 1883]

10. Find the value of $\dfrac{x^2+xy+y^2}{x^2-xy+y^2}$, when

$$x=\frac{\sqrt{a+1}+\sqrt{a-1}}{\sqrt{a+1}-\sqrt{a-1}}, \text{ and } y=\frac{\sqrt{a+1}-\sqrt{a-1}}{\sqrt{a+1}+\sqrt{a-1}}.$$

11. Find the value of $\left(\dfrac{x}{x-1}\right)^2+\left(\dfrac{x}{x+1}\right)^2$, when $x=\sqrt{\dfrac{n-1}{n+1}}$.

[Cal. 1885]

12. Extract the square roots of

(i) $1+\sqrt{1-m^2}$. (ii) $ax-2a\sqrt{ax-a^2}$.

(iii) $x+y+z+w+2\sqrt{xy+yz+zw+wx}$.

13. Express $xy+\sqrt{(x^2-1)(y^2-1)}$ in terms of a and b, when $2x=a+\dfrac{1}{a}$ and $2y=b+\dfrac{1}{b}$.

14. Show that $\sqrt{[a\sqrt{\{a\sqrt{(a\cdots\quad)\}}]}}$ to infinity $=a$.

15. Prove that the real value of $\sqrt[3]{(6\sqrt{3}+10)}-\sqrt[3]{(6\sqrt{3}-10)}$ is 2. [Bom. P. E. 1892]

16. If $x=2+2^{\frac{2}{3}}+2^{\frac{1}{3}}$, prove that $x^3-6x^2+6x-2=0$. [Cal. 1930]

17. Clear of radicals the equation $\sqrt{a}\pm\sqrt{b}\pm\sqrt{c}=0$.

18. If $a.\sqrt[3]{x^2}+b.\sqrt[3]{x}+c=0$, show that $a^3x^2+b^3x+c^3=3abcx$.

19. Show that if $\sqrt[3]{\{x^3+\sqrt{x^6-8y^6}\}}+\sqrt[3]{\{x^3-\sqrt{x^6-8y^6}\}}=a$, then $2x^3+6ay^2-a^3=0$.

20. Prove that $\sqrt{\left(\dfrac{x^3-a^3}{x^2-a^2}+3a\right)}=\dfrac{x+2a}{\sqrt{x+a}}$.

21. If $c=a\sqrt{1-b^2}+b\sqrt{1-a^2}$, prove that

$(a+b+c)(b+c-a)(c+a-b)(a+b-c)=4a^2b^2c^2$.

Simplify :

22. $\sqrt{\left\{\dfrac{2x^2+1+x\sqrt{4x^2+3}}{2x^2+3+x\sqrt{4x^2+3}}\right\}}$.

23. $\dfrac{\sqrt{ax}}{\sqrt{a}+\sqrt{x}-\sqrt{a+x}}-\dfrac{\sqrt{ax}}{\sqrt{a}+\sqrt{x}+\sqrt{a+x}}$. [Mad. 1875]

24. $\dfrac{\sqrt{1-x^2}+\sqrt{1-a^2}}{a\sqrt{1-x^2}+x\sqrt{1-a^2}}+\dfrac{a\sqrt{1-x^2}+x\sqrt{1-a^2}}{\sqrt{1-x^2}+\sqrt{1-a^2}}$.

25. $[\sqrt{1-a^2}.\sqrt{1-x^2}+ax]^2-2ax[\sqrt{1-a^2}.\sqrt{1-x^2}+ax]+x^2$. [Mad. 1869]

26. Show that $\dfrac{1}{\sqrt{x}-\sqrt{x-n}}+\dfrac{1}{\sqrt{x}+\sqrt{x+n}}=\dfrac{2}{\sqrt{x+n}-\sqrt{x-n}}$.

27. Show that $\dfrac{a\sqrt{a+x}}{\sqrt{a+x}-\sqrt{x}}=a+x+\sqrt{ax+x^2}$. [Bom. 1865]

28. If $\sqrt{\{(x-\sqrt{a^2-b^2})^2+y^2\}}+\sqrt{\{(x+\sqrt{a^2-b^2})^2+y^2\}}=2a$, show that $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$. [Mad. F.A. 1886]

CHAPTER V

RATIO AND PROPORTION

I. Ratio

42. Definitions.

When two quantities of the same kind are compared in respect of their magnitude, the number of times one quantity contains another is called the *ratio* of the first quantity to the second.

The ratio of any quantity a to any other quantity b of the same kind is usually written $a : b$. The quantities may be abstract or concrete, but the ratio is always a purely abstract number. The quantities a and b are called the *terms* of the ratio, a being the first term or *antecedent*, and b the second term or *consequent* of the ratio. Thus the ratio between 10 ft. and 3 ft. is expressed by $10 : 3$; that between 6 hours and 2 hours by $6 : 2$; and so on.

It is clear that two quantities of different kinds altogether cannot be compared with one another in respect of magnitude, and hence cannot bear any ratio between themselves. Thus, it is absurd to talk of any ratio between 7 miles and 3 hours and 5 mangoes.

The ratio of one quantity to another is therefore, determined by division, and hence can be expressed by a fraction; and therefore $a \div b$, $\frac{a}{b}$ and $a : b$ all represent the same algebraical operation, *viz.* division.

43. Equality of Ratio.

Since ratios can be represented by fractions, it is clear that *two ratios are equal if they are represented by equal fractions.*

Thus $6 : 8 = 12 : 16$, because the fractions representing the ratios are $\frac{6}{8}$ and $\frac{12}{16}$ respectively, and are obviously equal to one another,

We are thus led to the general

Theorem. *The value of a ratio remains unaltered if both the antecedent and the consequent are multiplied or divided by the same quantity.*

For, since $\frac{a}{b}=\frac{ma}{mb}$, and also $=\frac{a\div m}{b\div m}$,

it follows that $a : b=ma : mb$, and also $=\frac{a}{m} : \frac{b}{m}$.

It is thus clear that if two quantities are in the ratio of $a : b$, then we can write them as ak and bk, where k is any constant.

44. Ratios of Equality or Inequality.

A ratio is said to be a *ratio of equality* or *of greater inequality*, or *of less inequality*, according as the antecedent is equal to, or greater than, or less than the consequent, that is, according as the ratio is equal to, or greater than, or less than 1.

Thus $3 : 3$ and $a : a$ are ratios of equality, $7 : 5$ is a ratio of greater inequality, and $3 : 5$ is a ratio of less inequality; so too, $a : b$ is a ratio of greater inequality if $a>b$, and is a ratio of less inequality if $a<b$.

Concerning the ratios of inequality, we have the following important theorems:

(*i*) *A ratio of less inequality is increased, and a ratio of greater inequality is diminished, by adding the same quantity to both its terms.*

(*ii*) *A ratio of less inequality is diminished, and a ratio of greater inequality is increased, by subtracting the same quantity from both its terms.*

(i) Let $\frac{a}{b}$ be the ratio and let $\frac{a+x}{b+x}$ be the ratio formed by adding the same quantity x to both its terms.

Then, $\frac{a}{b}-\frac{a+x}{b+x}=\frac{x(a-b)}{b(b+x)}$ and therefore it is positive or or negative, according as $a-b$ is positive or negative, *i.e.*, according as $a>$ or $<b$.

Hence, if $a<b$, $\frac{x(a-b)}{b(b+x)}$ is negative, and $\frac{a+x}{b+x}>\frac{a}{b}$;

and if $a>b$, $\frac{x(a-b)}{b(b+x)}$ is positive and $\frac{a+x}{b+x}<\frac{a}{b}$;

which proves the proposition.

(ii) The proof is exactly similar.

45. Commensurable and Incommensurable Ratios and Quantities.

A ratio that can be exactly expressed as a ratio between two integers is called a *commensurable ratio*; and a ratio that cannot be so expressed is called an *incommensurable ratio*. Thus, since no two integers can be found whose ratio is exactly equal to $\sqrt{5} : 4$, or $\sqrt{3} : \sqrt{2}$ therefore $\sqrt{5} : 4$ or $\sqrt{3} : \sqrt{2}$ are incommensurable ratios, while $2 : 3$, $5\frac{1}{2} : 7\frac{3}{4}$, $\cdot 2\dot{2}\dot{5} : 3\dot{3}$ are all commensurable ratios, since they are equivalent to the ratios $2 : 3$, $22 : 31$ and $27 : 40$ respectively.

It is clear that a commensurable ratio can always be expressed in the form of a fraction or a decimal, terminating or recurring; while an incommensurable ratio can only be represented by a surd or a non-terminating non-recurring decimal. Two quantities are said to be *commensurable* or *incommensurable quantities* according as they bear to one another commensurable or incommensurable ratios.

46. Composition of Ratios.

(i) When two or more ratios are multiplied together, they are said to be *compounded*.

Thus, if the ratios $a : b$ and $c : d$ are compounded, we get

$$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd} = ac : bd.$$

Thus the ratio $ac : bd$ is compounded of the ratios $a : b$ and $c : d$.

Similarly, the ratio compounded of $a : b$, $c : d$ and $e : f$ is $ace : bdf$.

(ii) When two ratios are such that the antecedent of the one is the consequent of the other and *vice versa*, they are said to be *reciprocal* to each other.

Thus, the ratios $a : b$ and $b : a$ are reciprocal to each other. When they are compounded, the result is $ab : ba = 1 : 1 = 1$.

(iii) When the ratio $a : b$ is compounded with itself the result is the ratio $a^2 : b^2$, which is called the *duplicate* ratio of $a : b$. Similarly, $a^3 : b^3$ is the *triplicate* ratio, $a^4 : b^4$ the *quadruplicate* ratio of $a : b$; and so on.

Again, the ratio $a^{\frac{1}{2}} : b^{\frac{1}{2}}$ or $\sqrt{a} : \sqrt{b}$ is called the *sub-duplicate* ratio of $a : b$, and the ratio of $a^{\frac{1}{3}} : b^{\frac{1}{3}}$ or $\sqrt[3]{a} : \sqrt[3]{b}$ is the *sub-triplicate* ratio of $a : b$; and so on.

The ratio $a^{\frac{3}{2}} : b^{\frac{3}{2}}$ is called the *sesquiplicate* ratio of $a : b$.

Example 1. *Find the ratio compounded of $x-y : x+y$ and $x^2+xy : xy-y^2$.*

Since $\dfrac{x-y}{x+y} \times \dfrac{x^2+xy}{xy-y^2} = \dfrac{x-y}{x+y} \times \dfrac{x(x+y)}{y(x-y)} = \dfrac{x}{y}$,

the ratio compounded of the given ratios $= x : y$.

Example 2. *Compare the ratios $x^3+y^3 : x^2+y^2$ and $x^2+y^2 : x+y$, x and y being both positive.*

Since $\dfrac{x^3+y^3}{x^2+y^2} - \dfrac{x^2+y^2}{x+y} = \dfrac{xy^3+x^3y-2x^2y^2}{(x^2+y^2)(x+y)} = \dfrac{xy(x-y)^2}{(x^2+y^2)(x+y)}$,

which is a positive quantity, because $(x-y)^2$ is always positive whether $x-y$ is positive or negative, we have

$$\frac{x^3+y^3}{x^2+y^2} > \frac{x^2+y^2}{x+y}$$

i.e., $x^3+y^3 : x^2+y^2 > x^2+y^2 : x+y$.

Example 3. *Show that the ratio $ma+nc : mb+nd$ lies between the ratios $a : b$ and $c : d$, if these be unequal, all the quantities being positive.*

Of the unequal ratios $\dfrac{a}{b}$ and $\dfrac{c}{d}$, let $\dfrac{a}{b} > \dfrac{c}{d}$.

Multiplying both sides by bd, we have $ad > bc$.

Then $\dfrac{a}{b} - \dfrac{ma+nc}{mb+nd} = \dfrac{n(ad-bc)}{b(mb+nd)}$

which is positive, since $ad > bc$;

$$\therefore \qquad \frac{a}{b} > \frac{ma+nc}{mb+nd}.$$

Again, $\dfrac{ma+nc}{mb+nd} - \dfrac{c}{d} = \dfrac{m(ad-bc)}{d(mb+nd)}$,

which is positive, since $ad > bc$.

$$\therefore \qquad \frac{c}{d} < \frac{ma+nc}{mb+nd}.$$

Hence $\dfrac{ma+nc}{mb+nd}$ lies between $\dfrac{a}{b}$ and $\dfrac{c}{d}$.

Example 4. *If $a : b$ be in the duplicate ratio of $a+x : b+x$, find x.* [*Pun.* 1926]

We have $\dfrac{a}{b}=\left(\dfrac{a+x}{b+x}\right)^2=\dfrac{a^2+2ax+x^2}{b^2+2bx+x^2}$

$$\therefore \quad b(a^2+2ax+x^2)=a(b^2+2bx+x^2)$$

$$\text{or,} \quad x^2(a-b)=ab(a-b)$$

$$\text{or,} \quad x^2=ab$$

$$\therefore \quad x=\pm\sqrt{ab}.$$

Example 5. *Two numbers are in the ratio of 3 : 7, and if 13 be added to each, they are in the ratio of 7 : 12. Find the number.*

Let the numbers be $3x$ and $7x$ respectively so that they are in the ratio of 3 : 7.

Then, by the question, we have

$$\frac{3x+13}{7x+13}=\frac{7}{12}$$

$$\therefore \quad 36x+156=49x+91$$

$$\text{or,} \quad 13x=65$$

$$\therefore \quad x=5.$$

Hence the numbers are 15 and 35.

Example 6. *A certain ratio becomes 2 : 3 if 2 be subtracted from each of its terms, and becomes 3 : 4 if 3 be added to each of its terms. Find the ratio.*

Let $x : y$ be the ratio. Then, by the first condition,

$$\frac{x-2}{y-2}=\frac{2}{3} \quad \ldots \quad \ldots \quad (1)$$

and by the second condition,

$$\frac{x+3}{y+3}=\frac{3}{4} \quad \ldots \quad \ldots \quad (2)$$

From (1), and from (2), $\left.\begin{aligned} 3x-2y&=2 \\ 4x-3y&=-3 \end{aligned}\right\}$

Solving these simultaneous equations, we have $x=12$ and $y=17$. Hence the required ratio is 12 : 17.

Example 7. *What is the greatest positive integer which subtracted arithmetically from both the terms of the ratio* 13 : 17 *will give a ratio greater than* 2 : 5 ?

Let x be the greatest positive integer required. Then, by the question,

$$\frac{13-x}{17-x}-\frac{2}{5} \text{ must be positive ;}$$

i.e., $$\frac{31-3x}{5(17-x)} \text{ must be positive.}$$

Since, for arithmetical subtraction, x cannot exceed 13, therefore, in order that $\frac{31-3x}{5(17-x)}$ may be positive, 31 must be greater then $3x$, *i.e.*, x must be less than $10\frac{1}{3}$.

But the greatest integer less than $10\frac{1}{3}$ is 10.

Hence the required integer is 10.

EXERCISE 12.

Which is the greater ratio :

1. 3 : 4 or 5 : 7 ?
2. 6 : 11 or 8 : 15 ?
3. 14 : 17 or 11 : 13 ?
4. 5 : 9 or 7 : 11 ?
5. $x-2y : x+y$ or $x+3y : x+2y$?
6. $a+b : b$ or $4a : a+b$?
7. $a+b : a-b$ or $a^2+b^2 : a^2-b^2$, supposing $a > b$?
8. $x-4y : x-3y$ or $x-2y : x-y$?

Find the ratio compounded of :

9. 3 : 4, 6 : 7 and 8 : 9.
10. 3 : 7, 21 : 25 and 35 : 36.
11. $a^2 : bc$, $b^2 : ca$ and $c^2 : ab$.
12. $a+b : a-b$, $a^2-b^2 : (a+b)^2$ and $(a^2-b^2)^2 : a^4-b^4$.
13. $a^2-b^2 : a^2+b^2$, $a^3+b^3 : a^3-b^3$ and $a^2+ab+b^2 : a^2-ab+b^2$.

14. What number must be added to each term of the ratio 2 : 7 to make it equal to 3 : 4 ?

15. What number must be added to each term of the ratio 6 : 13 to make it equal to 2 : 3 ?

16. Find two numbers in the ratio $1\frac{1}{2} : 2\frac{2}{3}$ such that when increased by 15, they shall be in the ratio $1\frac{2}{3} : 2\frac{1}{2}$. [Pun. 1899]

17. A certain ratio becomes $2:3$ if 3 be added to each of its terms, and becomes $3:5$ if 4 be subtracted from each of its terms. Find the ratio.

18. Two numbers are in the ratio of $5:11$, and if 3 be added to each number they are in the ratio of $1:2$. Find the numbers.

19. What is the positive integer which when added to both the terms of the ratio $3:11$ will give a ratio just less than $4:7$?

20. Find the least positive integer which when added to both the terms of the ratio $4:9$ will give a ratio greater than $7:13$.

21. Two numbers are in the ratio $3:4$. If 5 be added to the greater and 8 taken from the less, the numbers are in the duplicate ratio of $5:7$. Find the numbers.

22. If $x-2y:2x+y=a:b$, find the value of $x:y$ in terms of a and b.

23. If $ax+by=bx+ay$, find the value of $x:y$.

24. If $2a-x:b-2x$ is in the sub-duplicate ratio of $a:b$, show that $x^2=ab$.

25. If $a-x:b-x$ is in the duplicate ratio of $a:b$, show that

$$\frac{1}{x}=\frac{1}{a}+\frac{1}{b}.$$

26. If a and b are unequal and $ab(c^2+d^2)=b^2c^2+a^2d^2$, show that $a:b$ is in the duplicate ratio of $c:d$.

27. If $p:q$ is in the duplicate ratio of $a+x:a-x$, show that

$$\frac{1}{2}\left(\frac{a}{x}+\frac{x}{a}\right)=\frac{p+q}{p-q}.$$

28. If $a+x:b+x$ be in the sub-triplicate ratio of $a:b$, prove that $x^3-3abx-ab(a+b)=0$.

29. What must be added to each of the terms of the ratio $a:b$, so that it may be equal to the ratio $c:d$?

30. If $x:y$ be the ratio $a:b$ in its lowest terms, prove that $\dfrac{x+1}{y+1}>\dfrac{a+1}{b+1}$, if $b>a$. [Cal. F. A. 1882]

31. What must be subtracted from each of the terms of the ratio $a:b$, so that the resulting ratio will be $2a:3b$?

32. If a, b, c, d are any four positive quantities, show that the ratio $a+c:b+d$ is intermediate between the ratios $a:b$ and $c:d$.

II. Proportion

47. Definitions.

When two ratios are equal, the four quantities composing them taken in order are called *proportionals*, and are said to be *in proportion*.

Thus, if $a : b = c : d$ then the four quantities a, b, c and d are proportionals. The proportion is written in two ways :

$$a : b = c : d,$$

or

$$a : b : : c : d.$$

of which the first is expressed by saying that "the ratio of a to b equals the ratio of c to d," or "a to b equals c to d," and the second is read "a is to b as c is to d."

The first and the last terms, a and d, are called the *extremes* and the other two, b and c, the *means* of the proportion.

When four quantities are in proportion the first two quantities may be of one kind, and the last two may be of a different kind. Thus, we may have the proportion

2 miles : 4 miles : : 5 horses : 10 horses.

Three or more quantities are said to be in *continued proportion*, when the first is to the second, as the second is to the third, as the third is to the fourth, and so on. Thus, a, b, c, d......are in continued proportion when

$$a : b = b : c = c : d = \cdots\cdots$$

When three quantities a, b, c are in continued proportion *i.e.*, when $a : b = b : c$, then b is called the *mean proportional* between a and c, and c is called the *third proportional* to a and b.

When a number of quantities are in continued proportion, it is clear that they must be all of the same kind.

48. Theorems on Proportionals.

If the proportionals are all abstract quantities, then the following theorems are true of them.

(*i*) *If four quantities are proportionals, the product of the extremes is equal to the product of the means.*

Let a, b, c, d be proportionals.

Then $\qquad a : b = c : d.$

i.e., $\frac{a}{b}=\frac{c}{d}$,

whence $ad=bc$. [by cross-multiplication]

Conversely, if $ad=bc$, then dividing both sides by bd

we have $\frac{a}{b}=\frac{c}{d}$,

i.e., $a:b=c:d$.

Hence, if any three terms of the proportion are given, the other may be found. Thus

$$a=\frac{bc}{d},\quad b=\frac{ad}{c},\quad c=\frac{ad}{b},\quad d=\frac{bc}{a}.$$

(*ii*) *If three quantities are in continued proportion, the product of the extremes is equal to the square of the mean.*

For, if $a:b::b:c$,

then $\frac{a}{b}=\frac{b}{c}$,

whence $ac=b^2$. [by cross-multiplication]

Thus $b=\sqrt{ac}$

i.e., $\sqrt{ac}$ is a mean proportional between a and c.

(*iii*) *If three quantities are in continued proportion, the first is to the third in the duplicate ratio of the first to the second.*

Let a, b, c be three quantities in continued proportion; then $a:b=b:c$

i.e., $\frac{a}{b}=\frac{b}{c}$.

Now $\frac{a}{c}=\frac{a}{b}\times\frac{b}{c}=\frac{a}{b}\times\frac{a}{b}=\frac{a^2}{b^2}$

that is, $a:c=a^2:b^2$,

which proves the proposition.

Similarly, if $a:b=b:c=c:d$, it can be easily proved that $a:d=a^3:b^3$, *i.e.*, $a:d$ is in the triplicate ratio of $a:b$; and so on.

Example 1. *Find a fourth proportional to* 14, 21 *and* 22

Let x be the fourth proportional.

Then we have $\quad 14 : 21 = 22 : x$

or, $$\frac{14}{21} = \frac{22}{x}$$

or, $$14x = 21 \times 22$$

$\therefore$ $$x = 33.$$

Example 2. *Find a third proportional to* 9 *and* 12.

Let x be the third propotional.

Then $$9 : 12 = 12 : x$$

or, $$\frac{9}{12} = \frac{12}{x}$$

$\therefore$ $$9x = 144$$

$\therefore$ $$x = 16.$$

Example 3. *Fin l a mean proportional between* 7 *and* 63.

Let x be the mean proportional.

Then $$7 : x = x : 63$$

or, $$x^2 = 7 \times 63 = 7 \times 7 \times 3 \times 3$$

$\therefore$ $$x = 7 \times 3 = 21.$$

Example 4. *If y is a mean proportional between z and x, show that $xy + yz$ is a mean proportional between $x^2 + y^2$ and $y^2 + z^2$.* [*Pun.* 1890]

By the question $\quad y^2 = zx.$

Now, $$(x^2 + y^2)(y^2 + z^2) = (x^2 + zx)(zx + z^2) \qquad [\because y^2 = zx]$$
$$= xz(x + z)(x + z)$$
$$= y^2(x + z)^2$$
$$= (xy + yz)^2,$$

which proves what is required.

EXERCISE 13.

1. Find a fourth proportional to :

(i) 6, 10, 15. (ii) 12, 21, 28. (iii) 16, 10, 24.

(iv) $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$. (v) $2, 3\frac{3}{4}, 2\frac{2}{3}$. (vi) $a^2bc, a^3b^2c^3$ and a^5bc^5.

2. Find a third proportional to :

(i) 2, 3. (ii) 8, 10. (iii) 9, 12. (iv) 16, 24.

(v) $\frac{9}{10}, \frac{3}{5}$. (vi) $a^3+b^3-ab(a+b)$ and a^2-b^2.

3. Find a mean proportional between :

(i) 9, 25. (ii) 3, 48. (iii) 18, 50. (iv) $1\frac{9}{16}, 1\frac{11}{25}$.

(v) $4x^2y, 9yz^2$. (vi) $\frac{p+q}{p-q}.a^2b$ and $\frac{p-q}{p+q}.ab^2$.

4. Find the value of x, when x is a mean proportional between $x-3$ and $x+8$.

5. If $a, 2a-b, a-2b$ and b be proportionals, prove that $a-b$ is the mean proportional between a and b.

6. If a, b, c are in continued proportion, prove that

(i) $\frac{1}{b^2}=\frac{1}{b^2-a^2}+\frac{1}{b^2-c^2}$.

(ii) $\left(\frac{a+b}{b+c}\right)^2=\frac{a^2+b^2}{b^2+c^2}$. [Bom. 1934]

(iii) $\frac{(a+b+c)^2}{a^2+b^2+c^2}=\frac{a+b+c}{a+b+c}$. [Pun. 1934]

7. If $a:b=c:d$, prove that $ab+cd$ is a mean proportional between a^2+c^2 and b^2+d^2. [Dac. 1926]

8. What must be subtracted from each of the numbers 17, 25, 31, 47, so that the remainders may be in proportion ?

9. If $\frac{x+y}{y+z}=\frac{x-y}{y-z}$, show that y is a mean proportional between x and z.

10. If $\frac{x^2+y^2}{y(z+x)}=\frac{x+y}{y+z}$, show that x, y, z are in continued proportion.

49. Derived Proportions.

If four quantities a, b, c, d are in proportion, many other proportions may be derived from them, of which the following are the most useful :

(*i*) *If* $a : b = c : d$, *then* $b : a = d : c$.

For,
$$\frac{a}{b} = \frac{c}{d}$$

$\therefore$
$$1 \div \frac{a}{b} = 1 \div \frac{c}{d};$$

whence
$$\frac{b}{a} = \frac{d}{c}$$

i.e.,
$$b : a = d : c.$$

This is called *Invertendo.*

(*ii*) *If* $a : b = c : d$, *then* $a : c = b : d$.

For.
$$\frac{a}{b} = \frac{c}{d}$$

$\therefore$
$$\frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}$$

or,
$$\frac{a}{c} = \frac{b}{d}$$

i.e.,
$$a : c = b : d$$

This is called *Alternando.*

(*iii*) *If* $a : b = c : d$, *then* $a + b : b = c + d : d$.

For.
$$\frac{a}{b} = \frac{c}{d}$$

$\therefore$
$$\frac{a}{b} + 1 = \frac{c}{d} + 1$$

or,
$$\frac{a+b}{b} = \frac{c+d}{d}$$

i.e.,
$$a + b : b = c + d : d.$$

This is called *Componendo.*

(*iv*) *If* $a : b = c : d$, *then* $a - b : b = c - d : d$.

For,
$$\frac{a}{b} = \frac{c}{d}$$

$\therefore$
$$\frac{a}{b} - 1 = \frac{c}{d} - 1$$

or,
$$\frac{a-b}{b} = \frac{c-d}{d}$$

i.e.,
$$a - b : b = c - d : d.$$

This is called *Dividendo.*

(*v*) *If* $a : b = c : d$, *then* $a+b : a-b = c+d : c-d$.

For, by (iii), $$\frac{a+b}{b} = \frac{c+d}{d},$$

and by (iv), $$\frac{a-b}{b} = \frac{c-d}{d}.$$

$\therefore$ by division, $$\frac{a+b}{a-b} = \frac{c+d}{c-d},$$

i.e., $$a+b : a-b = c+d : c-d.$$

This is called *Componendo and Dividendo.*

(*vi*) *If* $a : b = c : d$, *then* $a : a-b = c : c-d$.

For, by (iv), $$\frac{a-b}{b} = \frac{c-d}{d}.$$

Hence by (i), $$\frac{b}{a-b} = \frac{d}{c-d}.$$

$\therefore$ since $$\frac{a}{b} = \frac{c}{d}$$

we have $$\frac{b}{a-b} \times \frac{a}{b} = \frac{d}{c-d} \times \frac{c}{d}$$

or, $$\frac{a}{a-b} = \frac{c}{c-d},$$

i. e., $$a : a-b = c : c-d.$$

This is called *Convertendo.*

50. An Important Theorem

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \ldots\ldots\ldots\ldots$, *then each of these ratios*

$$= \left(\frac{pa^n + qc^n + re^n + \cdots\cdots}{pb^n + qd^n + rf^n + \cdots\cdots}\right)^{\frac{1}{n}},$$

where $p, q, r \cdots\cdots\cdots n$ *are any quantities whatsoever.*

Let $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \cdots\cdots\cdots\cdots = k\,;$$

then $$a = bk, \quad c = dk, \quad e = fk, \cdots\cdots\cdots;$$

whence, $$pa^n = p(bk)^n = pb^n k^n,$$
$$qc^n = q(dk)^n = qd^n k^n,$$
$$re^n = r(fk)^n = rf^n k^n \text{ ; and so on.}$$

$\therefore$ Hence by addition, we have
$$pa^n + qc^n + re^n + \cdots\cdots = (pb^n + qd^n + rf^n + \cdots\cdots\cdots)k^n$$

or, $$k^n = \frac{pa^n + qc^n + re^n + \dots\dots\dots}{pb^n + qd^n + rf^n + \dots\dots\dots}$$

$\therefore$ $$k = \left(\frac{pa^n + qc^n + re^n + \dots\dots}{pb^n + qd^n + rf^n + \dots\dots}\right)^{\frac{1}{n}}$$

which proves the proposition.

By giving different values to $p, q, r \dots\dots n$, many particular cases of this general theorem may be deduced. It is, however, advisable to prove them independently by the same method. Thus,

(i) putting $n=1$, each of the ratios

$$= \frac{pa + qc + re + \dots}{pb + qd + rf + \dots}$$

(ii) putting $p=q=r=n=1$, each of the ratios

$$= \frac{a+c+e+\dots}{b+d+f+\dots}$$

i.e., when a series of ratios are equal, each of them is equal to the ratio of the sum of all the antecedents to the sum of all the consequents.

Example 1. *If $a : b = c : d$, show that*

$$ma + nc : mb + nd = \sqrt{(pa^2 + qc^2)} : \sqrt{(pb^2 + qd^2)}.$$

Let $\frac{a}{b} = \frac{c}{d} = k$; then $a = bk$ and $c = dk$.

$\therefore$ $ma + nc = mbk + ndk = (mb + nd)k$

i.e., $$\frac{ma + nc}{mb + nd} = k \quad \dots \quad \dots \quad (1)$$

Again, $a^2 = b^2k^2$, and $c^2 = d^2k^2$.

$\therefore$ $pa^2 + qc^2 = pb^2k^2 + qd^2k^2 = (pb^2 + qd^2)k^2$;

or $\sqrt{(pa^2 + qc^2)} = \sqrt{(pb^2 + qd^2)}k$,

i.e., $$\frac{\sqrt{(pa^2 + qc^2)}}{\sqrt{(pb^2 + qd^2)}} = k \quad \dots \quad \dots \quad (2)$$

Hence, from (1) and (2),

$$\frac{ma + nc}{mb + nd} = \frac{\sqrt{(pa^2 + qc^2)}}{\sqrt{(pb^2 + qd^2)}},$$

each being equal to k.

Example 2. *If $a : b = c : d$, prove that*

$$a^2+ab+b^2 : a^2-ab+b^2 = c^2+cd+d^2 : c^2-cd+d^2.$$

[*Cal.* 1894 & *Dac.* 1935]

Since $\frac{a}{b}=\frac{c}{d}$, $\therefore \frac{a}{c}=\frac{b}{d}=k$ (suppose) [by *Alternando*]

Hence $a=ck$, and $b=dk$.

$\therefore \quad a^2=c^2k^2,\ ab=cdk^2,\ b^2=d^2k^2.$

$\therefore \quad a^2+ab+b^2=c^2k^2+cdk^2+d^2k^2=(c^2+cd+d^2)k^2$;

and $\quad a^2-ab+b^2=c^2k^2-cdk^2+d^2k^2=(c^2-cd+d^2)k^2$.

Hence, by division, $\frac{a^2+ab+b^2}{a^2-ab+b^2}=\frac{c^2+cd+d^2}{c^2-cd+d^2}$.

Example 3. *If* $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$, *prove that*

$$\frac{x^3}{a^2}+\frac{y^3}{b^2}+\frac{z^3}{c^2}=\frac{(x+y+z)^3}{(a+b+c)^2}.$$

[*Cal.* 1901]

Let each of the ratios be $=k$; then $x=ak$, $y=bk$ and $z=ck$.

$$\therefore \quad \frac{x^3}{a^2}=\frac{a^3k^3}{a^2}=ak^3,$$

$$\frac{y^3}{b^2}=\frac{b^3k^3}{b^2}=bk^3,$$

and

$$\frac{z^3}{c^2}=\frac{c^3k^3}{c^2}=ck^3.$$

$$\therefore \quad \frac{x^3}{a^2}+\frac{y^3}{b^2}+\frac{z^3}{c^2}=(a+b+c)k^3 \quad \dots \quad \dots \quad (1)$$

Again, $\quad x+y+z=(a+b+c)k$

$\therefore \quad (x+y+z)^3=(a+b+c)^3k^3$

or, $$\frac{(x+y+z)^3}{(a+b+c)^2}=(a+b+c)k^3 \quad \dots \quad \dots \quad (2)$$

Hence, from (1) and (2),

$$\therefore \quad \frac{x^3}{a^2}+\frac{y^3}{b^2}+\frac{z^3}{c^2}=\frac{(x+y+z)^3}{(a+b+c)^2}$$

each being equal to $(a+b+c)k^3$.

Example 4. *If a, b, c, d be in continued proportion, prove that* $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$. [Pat. 1924]

Since $\dfrac{a}{b}=\dfrac{b}{c}=\dfrac{c}{d}$.

each of these ratios $=\dfrac{a^2}{ab}=\dfrac{b^2}{bc}=\dfrac{c^2}{cd}=\dfrac{a^2+b^2+c^2}{ab+bc+cd}$ ... (1)

Also each of these ratios $=\dfrac{ab}{b^2}=\dfrac{bc}{c^2}=\dfrac{cd}{d^2}=\dfrac{ab+bc+cd}{b^2+c^2+d^2}$ ... (2)

Hence, from (1) and (2), $\dfrac{a^2+b^2+c^2}{ab+bc+cd}=\dfrac{ab+bc+cd}{b^2+c^2+d^2}$.

$\therefore \quad (a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$.

EXERCISE 14.

1. If $a : b=c : d$, prove that

(i) $a-b : c-d=a+c : c+d$. [Bom. 1927]

(ii) $a^2+b^2 : a^2-b^2=ac+bd : ac-bd$. [Dac. 1927]

(iii) $ma-nb : a+b=mc-nd : c+d$. [Cal. 1893]

(iv) $ac : bd=a^2+c^2 : b^2+d^2$. [Cal. 1877]

(v) $(a+c)^3 : (b+d)^3=a(a-c)^2 : b(b-d)^2$. [Cal. 1888]

(vi) $\sqrt{(a^2+c^2)} : \sqrt{(b^2+d^2)}=ma+nc : mb+nd$. [Cal. 1880]

(vii) $pa^2+qc^2 : pb^2+qd^2=ma^2-nc^2 : mb^2-nd^2$. [Cal. 1899]

(viii) $\sqrt[n]{a}+\sqrt[n]{b} : \sqrt[n]{c}+\sqrt[n]{d}=\sqrt[n]{(a+b)} : \sqrt[n]{(c+d)}$.

2. If $a : b=c : d=e : f$, prove that each of these ratios is equal to :

(i) $a-2c+3e : b-2d+3f$.

(ii) $ka+lc+me : kb+ld+mf$. [Cal. 1875]

(iii) $\sqrt{(3a^2-2c^2+5e^2)} : \sqrt{(3b^2-2d^2+5f^2)}$.

(iv) $\sqrt[3]{(a^3+c^3+e^3)} : \sqrt[3]{(b^3+d^3+f^3)}$. [Pat. 1930]

3. If $a : b=c : d=e : f$, prove that

(i) $a+c+e : pa+qc+re=b+d+f : pb+qd+rf$.

(ii) $2a^3-3c^3+4e^3 : 2b^3-3d^3+4f^3=a^2e : b^2f$. [Pat. 1932]

(iii) $a^2+c^2+e^2 : b^2+d^2+f^2=(a+c+e)^2 : (b+d+f)^2$.

(iv) $(a^2+c^2+e^2)(b^2+d^2+f^2)=(ab+cd+ef)^2$. [Pat. 1931]

(v) $(a+c+e)^3 : (b+d+f)^3 = a(a-c)(a-e) : b(b-d)(b-f)$.

(vi) $a^2ce : b^2df = \sqrt[4]{(a^5c^3e^8)} : \sqrt[4]{(b^5d^3f^8)}$. [Pun. 1933]

(vii) $(a^2+b^2)(ce+df)^2 = (c^2+d^2)(ae+bf)^2 = (e^2+f^2)(ac+bd)^2$

(viii) $\sqrt{(a+c+e)}\sqrt{(b+d+f)} = \sqrt{ab} + \sqrt{cd} + \sqrt{ef}$.

4. If a, b, c, d be in continued proportion, show that

(i) $b+c : c-d = a+b : b-c$.

(ii) $a : d = a^3+b^3+c^3 : b^3+c^3+d^3$. [Cal. 1934 & Dac.1935]

(iii) $a : d = pa^3+qb^3+rc^3 : pb^3+qc^3+rd^3$. [Cal. 1898]

(iv) $a+b : c+d = pa^2+qb^2+rc^2 : pb^2+qc^2+rd^2$.

(v) $a^2-ab+b^2 : a^2+ab+b^2 = b^2-bc+c^2 : b^2+bc+c^2$.

(vi) $(b-c)^2+(c-a)^2+(d-b)^2 = (a-d)^2$. [Dac. 1933]

5. If $c : d = x : y$, then $cd : xy = c^2+d^2 : x^2+y^2$. [Pun.1892]

6. If $x : a = y : b = z : c$, prove that

(i) $\dfrac{x^3}{a^3} + \dfrac{y^3}{b^3} + \dfrac{z^3}{c^3} = 3\left(\dfrac{x+y+z}{a+b+c}\right)^3$. [Dac. 1930]

(ii) $\left(\dfrac{a^2x^2+b^2y^2+c^2z^2}{a^3x+b^3y+c^3z}\right)^{\frac{3}{2}} = \sqrt{\dfrac{xyz}{abc}}$ [Pun. 1929]

7. If $a : b = b : c$, show that

$$a^2b^2c^2\left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right) = a^3+b^3+c^3.$$

8. If $x : a = y : b = z : c$, prove that

$$\frac{x^2+y^2}{a^2+b^2} = \frac{y^2+z^2}{b^2+c^2} = \frac{z^2+x^2}{c^2+a^2} = \left(\frac{lx+my+nz}{la+mb+nc}\right)^2.$$

[Cal. F. A. 1869]

9. If $\dfrac{x_1}{x_2} = \dfrac{x_2}{x_3} = \dfrac{x_3}{x_4} = \dots\dots = \dfrac{x_n}{x_{n+1}}$, prove that

$$\left(\frac{x_1+x_2+x_3+\dots\dots+x_n}{x_2+x_3+x_4+\dots\dots+x_{n+1}}\right)^n = \frac{x_1}{x_{n+1}}.$$

10. If $\dfrac{a}{b}, \dfrac{c}{d}, \dfrac{e}{f}, \dots\dots$ be a number of unequal ratios with $a, b, c, d, e, f, \dots\dots$ all positive, show that the ratio

$$\frac{a+c+e+\dots\dots}{b+d+f+\dots\dots}$$

lies between the greatest and the least of these ratios.

51. Various types of examples involving ratio and proportion are worked out below.

Example 1. *If* $\dfrac{a^2+ay+y^2}{b^2+bx+x^2}=\dfrac{a^2-ay+y^2}{b^2-bx+x^2}$, *prove that either* $x:a=b:y$, or $x:b=y:a$.

We have, by *Alternando*,

$$\frac{a^2+ay+y^2}{a^2-ay+y^2}=\frac{b^2+bx+x^2}{b^2-bx+x^2}$$

$\therefore$ $\dfrac{a^2+y^2}{ay}=\dfrac{b^2+x^2}{bx}$ [by *Componendo* and *Dividendo*.]

or, $bx(a^2+y^2)-ay(b^2+x^2)=0$,

or, $a^2bx+bxy^2-ab^2y-ax^2y=0$,

or, $xy(by-ax)-ab(by-ax)=0$,

or, $(xy-ab)(by-ax)=0$,

$\therefore$ either $xy-ab=0$, whence $x:a=b:y$,

or, $by-ax=0$, whence $x:b=y:a$.

Example 2. *If* $x(b-c)+y(c-a)+z(a-b)=0$, *prove that*

$$\frac{b-c}{bz-cy}=\frac{c-a}{cx-az}=\frac{a-b}{ay-bx}.$$

We have $x(b-c)+y(c-a)+z(a-b)=0$ ... ... (1)

Also, $a(b-c)+b(c-a)+c(a-b)=0$, identically ... (2)

From (1) and (2), by the rule of cross-multiplication, we have

$$\frac{b-c}{cy-bz}=\frac{c-a}{az-cx}=\frac{a-b}{bx-ay}$$

$$\frac{b-c}{bz-cy}=\frac{c-a}{cx-az}=\frac{a-b}{ay-bx}.$$

Example 3. *If* $\dfrac{x}{bc(b+c)}=\dfrac{y}{ca(c+a)}=\dfrac{z}{ab(a+b)}$, *prove that*

$$\frac{b+c}{x(by+cz-ax)}=\frac{c+a}{y(cz+ax-by)}=\frac{a+b}{z(ax+by-cz)}.$$

Let each of the given ratios $=k$; then

$$k.abc=\frac{ax}{b+c}=\frac{by}{c+a}=\frac{cz}{a+b}=\frac{by+cz-ax}{(c+a)+(a+b)-(b+c)}$$

$$=\frac{by+cz-ax}{2a} \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

Also $$k=\frac{x}{bc(b+c)} \quad \cdots \quad \cdots \quad (2)$$

$$\therefore \quad k^2abc=\frac{by+cz-ax}{2a}\times\frac{x}{bc(b+c)}=\frac{x(by+cz-ax)}{2abc(b+c)}$$

$$2k^2a^2b^2c^2=\frac{x(by+cz-ax)}{b+c}.$$

Similarly, it can be shown that

$$2k^2a^2b^2c^2=\frac{y(cz+ax-by)}{c+a}=\frac{z(ax+by-cz)}{a+b}.$$

Hence, $$\frac{b+c}{x(by+cz-ax)}=\frac{c+a}{y(cz+ax-by)}=\frac{a+b}{z(ax+by-cz)}.$$

Example 4. *If* $\dfrac{2x+3y}{a+2b}=\dfrac{2y+3z}{b+2c}=\dfrac{2z+3x}{c+2a}$, *show that*

$$\frac{11x+7y+12z}{7a+4b+7c}=\frac{16x+12y+17z}{10a+7b+10c}.$$

Let each of the given ratios $=k$; then

$$k=\frac{l(2x+3y)}{l(a+2b)}=\frac{m(2y+3z)}{m(b+2c)}=\frac{n(2z+3x)}{n(c+2a)}$$

$$=\frac{l(2x+3y)+m(2y+3z)+n(2z+3x)}{l(a+2b)+m(b+2c)+n(c+2a)}$$

$$=\frac{(2l+3n)x+(2m+3l)y+(2n+3m)z}{(l+2n)a+(m+2l)b+(n+2m)c}.$$

Now, let l, m, n be such that $2l+3n=11$, $2m+3l=7$ and $2n+3m=12$, so that the numerator becomes $11x+7y+12z$.

Solving these equations we have $l=1$, $m=2$, $n=3$.

$$\therefore \quad k=\frac{11x+7y+12z}{(1+6)a+(2+2)b+(3+4)c}=\frac{11x+7y+12z}{7a+4b+7c} \quad \cdots \quad (1)$$

Again, supposing $2l+3n=16$, $2m+3l=12$ and $2n+3m=17$, so that the numerator becomes $16x+12y+17z$, we have $l=2$, $m=3$, $n=4$.

$$\therefore \quad k=\frac{16x+12y+17z}{(2+8)a+(3+4)b+(4+6)c}=\frac{16x+12y+17z}{10a+7b+10c} \quad \cdots \quad (2)$$

Hence, from (1) and (2),

$$\frac{11x+7y+12z}{7a+4b+7c}=\frac{16x+12y+17z}{10a++7b+10c},$$

each of them being equal to k.

EXERCISE 15.

1. If $\frac{x}{b-c}=\frac{y}{c-a}=\frac{z}{a-b}$, find the value of

$(b+c-a)x+(c+a-b)y+(a+b-c)z$. [Pat. 1930]

2. If $\frac{bz-cy}{a}=\frac{cx-az}{b}=\frac{ay-bx}{c}$, show that $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

3. If $x:a=y:b=z:c$, show that

(i) $\frac{ax-by}{x^2-y^2}=\frac{by-cz}{y^2-z^2}=\frac{cz-ax}{z^2-x^2}$.

(ii) $\frac{ax-by}{(a-b)(x+y)}=\frac{by-cz}{(b-c)(y+z)}=\frac{cz-ax}{(c-a)(z+x)}$.

4. If $x=\frac{\sqrt{(a+2b)}+\sqrt{(a-2b)}}{\sqrt{(a+2b)}-\sqrt{(a-2b)}}$, prove that $bx^2-ax+b=0$.

[Cal. Int. 1935]

5. If $\frac{a}{b+c}=\frac{b}{c+a}=\frac{c}{a+b}$, prove that each of these fractions is equal to $\frac{1}{2}$ or -1. [Bom. P. E. 1895]

6. If $\frac{x}{(b-c)(b+c-2a)}=\frac{y}{(c-a)(c+a-2b)}=\frac{z}{(a-b)(a+b-2c)}$, prove that $x+y+z=0$. [Cal. 1889]

7. If $x(b-c)+y(c-a)+z(a-b)=0$, prove that

$$\frac{b-c}{y-z}=\frac{c-a}{z-x}=\frac{a-b}{x-y}.$$

8. If $\frac{a}{b+c-a}=\frac{b}{c+a-b}=\frac{c}{a+b-c}$, and $a+b+c$ be not zero, then $a=b=c$.

9. If $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$, then

$(a+b+c)(yz+zx+xy)=(x+y+z)(ax+by+cz)$. [Cal. Int. 1910]

10. If $\frac{l(mb-nc)}{x}=\frac{m(nc-la)}{y}=\frac{n(la-mb)}{z}$, then

$$\frac{ax}{by+cz}=\frac{by}{cz+ax}=\frac{cz}{ax+by}.$$

11. Show that $a : b = c : d$, if
$$(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d).$$
[Pat. 1931]

12. If $y+z : z+x : x+y = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}$, prove that
$$\frac{b-c}{y^2-z^2}=\frac{c-a}{z^2-x^2}=\frac{a-b}{x^2-y^2}.$$

13. If $a : y+z = b : z+x = c : x+y$, prove that
$$\frac{a(b-c)}{y^2-z^2}=\frac{b(c-a)}{z^2-x^2}=\frac{c(a-b)}{x^2-y^2}.$$
[Dac. 1931]

14. If $\frac{x+y}{a+b}=\frac{y+z}{b+c}=\frac{z+x}{c+a}$, then $\frac{4x+2y+3z}{4a+2b+3c}=\frac{5x+6y+4z}{5a+6b+4c}$.

15. If $\frac{b-c}{bz+cy}=\frac{c-a}{cx+az}=\frac{a-b}{ay+bx}$, show that
$$(a+b+c)(x+y+z)=ax+by+cz.$$

16. If $\frac{x}{b+c-a}=\frac{y}{c+a-b}=\frac{z}{a+b-c}$, find the value of
$$(b-c)x+(c-a)y+(a-b)z.$$
[Pat. 1918]

17. If $\frac{x}{1-x^2}=\frac{y+z}{1+yz}$, and $\frac{y}{1-y^2}=\frac{z+x}{1+zx}$, x and y being unequal, prove that $\frac{z}{1-z^2}=\frac{x+y}{1+xy}$.

18. If $a+b : b+c = c+d : d+a$, prove that either $c=a$ or $a+b+c+d=0$. [Cal. 1891]

19. If $a(y+z-x)=b(z+x-y)=c(x+y-z)$, prove that
$$\frac{x}{a(b+c)}=\frac{y}{b(c+a)}=\frac{z}{c(a+b)}.$$
[Mad. F. A. 1893]

20. If $\frac{b+c-a}{y+z-x}=\frac{c+a-b}{z+x-y}=\frac{a+b-c}{x+y-z}$, show that
$$x : y : z = a : b : c.$$

21. If $\frac{x}{y+z}=a$, $\frac{y}{z+x}=b$, $\frac{z}{x+y}=c$, show that
$$\frac{x^2}{a(1-bc)}=\frac{y^2}{b(1-ca)}=\frac{z^2}{c(1-ab)}.$$

22. If $(2am+6bm+3cn+9dn)(2am-6bm-3cn+9dn)$
$=(2am-6bm+3cn-9dn)(2am+6bm-3cn-9dn)$,
find the simplest relation connecting a, b, c, d. [Cal. Int 1911]

23. If $a:b=x-2y:y+2x$, show that $x:y=a+2b:b-2a$.

24. If $b+c$ is a mean proportional between $a+b$ and $c+a$, show that $b+c:c+a=c-a:a-b$.

25. If $\frac{x+y}{3a-b}=\frac{y+z}{3b-c}=\frac{z+x}{3c-a}$, then $\frac{x+y+z}{a+b+c}=\frac{ax+by+cz}{a^2+b^2+c^2}$

26. If $x+y+z=0$, and $\frac{x^2}{b-c}+\frac{y^2}{c-a}+\frac{z^2}{a-b}=0$, show that

$$\frac{x}{b-c}=\frac{y}{c-a}=\frac{z}{a-b}.$$

27. If $\frac{y+z}{pb+qc}=\frac{z+x}{pc+qa}=\frac{x+y}{pa+qb}$ show that

$$\frac{2(x+y+z)}{a+b+c}=\frac{(b+c)x+(c+a)y+(a+b)z}{bc+ca+ab}.$$

28. If $a+\frac{y^2-z^2}{b-c}=b+\frac{z^2-x^2}{c-a}$, show that each of them

$$=c+\frac{x^2-y^2}{a-b}.$$

29. If $\frac{3x+2y}{3a-2b}=\frac{3y+2z}{3b-2c}=\frac{3z+2x}{3c-2a}$, then will

$5(x+y+z)(5c+4b-3a)=(9x+8y+13z)(a+b+c)$.
[Cal. Int. 1912]

30. If $\frac{y}{z}-\frac{z}{y}=\frac{b-c}{x}$ and $\frac{z}{x}-\frac{x}{z}=\frac{c-a}{y}$, prove that

$$\frac{x}{y}-\frac{y}{x}=\frac{a-b}{z}.$$ [Pun. 1888]

31. A's present age is to B's present age as $8:7$; 27 years ago their ages were as $5:4$. Find their present ages.
[All. 1900]

32. Two armies number 11000 men and 7000 men respectively; before they fight, each is re-inforced by 1000 men. In favour of which army is the increase? [Cal. 1879]

CHAPTER VI

VARIATION.

52. Definitions.

If we consider any function of x, say, x^2+3x+4, and denote it by y, we see that if x changes or varies, the value of the function y also changes and varies. In fact, this is the meaning of y being a function of x ; and in such cases, x is called the *independent variable* and the function y the *dependent variable*. [See Art. 2.]

If, however, the function is such that the change in y is proportional to the change in x, then y is said to *vary directly* as x, or simply, to *vary as* x. Thus, if $y=mx$, where m is a constant, so that if $x=a$, $y=ma$, if $x=b$, $y=mb$, if $x=c$, $y=mc$, and the changes are evidently proportional, for $ma : mb : mc = a : b : c$, then y is said to vary directly as x, or simply, to vary as x.

This is the restricted sense in which the word *variation* is ordinarily used in Algebra.

When y varies as x, it is written thus $\propto x$, the symbol $\propto$ denoting direct variation.

53. Direct Variation.

It follows from the definition given above that *when y varies directly as x, if y_1, y_2 are two values of y corresponding to two values x_1, x_2 of x, then*

$$y_1 : y_2 = x_1 : x_2 ;$$

for direct variation means that the changes are always proportional.

Again, *when y varies directly as x, then $y=mx$, where m is a constant.*

This constant is called the *constant of variation.*

The constant of variation can be determined if any pair of corresponding values of the variables are given ; *e.g.*, if $y \propto x$, and $y=3$ when $x=5$, we have from the relation $y=mx$, $3=m.5$, or $m=\frac{3}{5}$, and so the relation between y and x is $y=\frac{3}{5}x$.

Examples of direct variation are easy to find. If a person travels at a uniform rate, say, at 4 miles per hour, then he will travel 8 miles in 2 hours, 12 miles in 3 hours, and so on. This result can be expressed by saying that the distance travelled varies directly as the time taken. In symbols, we can write it as $s \propto t$, where s stands for the distance and t the time.

Similarly, the circumference of a circle is always proportional to the radius: hence we may say that the circumference varies as the radius.

Again, the area of a triangle of given base varies as the altitude; and the area of a triangle of given altitude varies as the base; for we know that half the product of the base and the altitude measures the area of a triangle.

54. Inverse Variation.

A quantity y is said to *vary inversely as* x, when y varies directly as the reciprocal of x.

Thus, y varies inversely as x, if $y \propto \frac{1}{x}$, *i.e.*, if $y = \frac{m}{x}$.

It follows therefore that $xy = m$, a constant. This leads to another definition of inverse variation; viz., *when two quantities are so related that their product is constant, either of them is said to vary inversely as the other.*

Examples of inverse variation can also be readily given. If the distance to be travelled is fixed, the time of travel and the rate of travel vary inversely as one another. Again, if the area of a triangle is given, the base varies inversely as the altitude and *vice versa*.

It is clear that in all cases of inverse variation, if either of the variables increases, the other diminishes in the same ratio and *vice versa*.

55. Joint Variation.

One quantity is said to *vary jointly* as a number of other quantities when it varies directly as their product.

Thus, u varies jointly as x, y, and z, if $u \propto xyz$, *i.e.*, if $u = mxyz$, where m is a constant.

For example, the area of a triangle varies jointly as the base and the altitude; the interest on a sum of money varies jointly as the sum of money invested, the time and the rate of interest; and so on.

If a quantity varies *jointly* as a second quantity and the reciprocal of a third, then it is said to *vary directly* as the second quantity and *inversely* as the third; *e. g.*, if $y = m.\frac{x}{z}$, then y is said to vary directly as x, and inversely as z.

For instance, the duration of a journey varies directly as the distance travelled and inversely as the speed; the acceleration of a body varies directly as the moving force and inversely as the mass of the body moved.

56. A few important results.

(*i*) *If $A \propto B$, and $B \propto C$, then $A \propto C$.*

Since $A \propto B$, we have $A = mB$, where m is constant, and since $B \propto C$, we have $B = nC$, where n is constant.

$$\therefore \qquad A = mn.C.$$

Hence $A \propto C$, for mn is a constant.

(*ii*) If $A \propto B$, *then* $A^n \propto B^n$.

Since $A \propto B$, we have $A = mB$, where m is constant,

$\therefore \qquad A^n = (mB)^n = m^n B^n$, where m^n is constant.

Hence $\qquad A^n \propto B^n$.

(*iii*) *If $A \propto C$ and $B \propto C$, then $A \pm B \propto C$ and $\sqrt{AB} \propto C$.*

Since $A = mC$, and $B = nC$, m and n being constants, we have, by addition and subtraction,

$A \pm B = (m \pm n)C$, where $m \pm n$ is constant.

Hence $\qquad A \pm B \propto C$.

Again $\qquad AB = mnC^2$.

$\therefore \qquad \sqrt{AB} = \sqrt{mn}.C$, where $\sqrt{mn}$ is constant.

Hence $\qquad \sqrt{AB} \propto C$.

(*iv*) *If $A \propto B$ and $C \propto D$, then $AC \propto BD$ and $\frac{A}{C} \propto \frac{B}{D}$.*

Since $A = mB$, and $C = nD$, m and n being constants, we have $\quad AC = mnBD$, and $\frac{A}{C} = \frac{m}{n} \cdot \frac{B}{D}$.

Hence $AC \propto BD$, and $\frac{A}{C} \propto \frac{B}{D}$.

(v) *If* $A \propto BC$, *then* $B \propto \frac{A}{C}$ *and* $C \propto \frac{A}{B}$.

Since $A = mBC$, m being constant, we have

$$B = \frac{1}{m}\cdot\frac{A}{C}, \text{ and } C = \frac{1}{m}\cdot\frac{A}{B}.$$

Hence $\qquad B \propto \frac{A}{C}$ and $C \propto \frac{A}{B}$.

57. Variations of Homogeneous Functions.

It is evident that if $x \propto y$, any homogeneous expression in x and y varies as any other homogeneous expression in x and y of the same degree. And this property holds good for homogeneous functions of any number of variables, provided that all the variables vary as one another.

This is illustrated in the following examples.

1. *If* $x \propto y$, *prove that* $x^3 + y^3 \propto x^2y - xy^2$.

Since $x \propto y$, we have $x = my$.
Hence $x^3 + y^3 = m^3y^3 + y^3 = (m^3 + 1)y^3$;
and $x^2y - xy^2 = m^2y^3 - my^3 = (m^2 - m)y^3$

$$\therefore \qquad \frac{x^3 + y^3}{x^2y - xy^2} = \frac{m^3 + 1}{m^2 - m}, \text{ constant.}$$

Hence, $\qquad x^3 + y^3 \propto x^2y - xy^2$.

Example 2. *If* $x \propto y \propto z$, *prove that*

$$ax^2 + by^2 + cz^2 \propto pyz + qzx + rxy.$$

Since $x \propto z$, we have $x = mz$; and since $y \propto z$, $y = nz$.

$\therefore\ ax^2 + by^2 + cz^2 = am^2z^2 + bn^2z^2 + cz^2 = (am^2 + bn^2 + c)z^2$;

$pyz + qzx + rxy = pnz^2 + qmz^2 + rmnz^2 = (pn + qm + rmn)z^2$.

Hence, $\dfrac{ax^2 + by^2 + cz^2}{pyz + qzx + rxy} = \dfrac{am^2 + bn^2 + c}{pn + qm + rmn}$, a constant.

$\therefore \qquad ax^2 + by^2 + cz^2 \propto pyz + qzx + rxy$.

58. Theorem on Joint Variation.

If $A \propto B$ *when* C *is constant, and* $A \propto C$ *when* B *is constant, then* $A \propto BC$ *when both* B *and* C *vary.* 9

The variation of A depends partly on the variation of B and partly on that of C, the variation of B being independent of that of C. Since the variation of B is quite independent of that of C, when both B and C vary, each of them produces its own effect independently on A. Let a be the value of A

when B changes to b and C to c, so that the change of A to a is due to two simultaneous changes, namely the change of B to b and that of C to c.

If C does not undergo any change, but remains constant while B changes to b, then A cannot change to a but it will undergo a different change and will assume some value a' different from a. Thus the change of A to a' is due to the change of B to b only, C remaining constant; and since $A \propto B$, we have by definition,

$$\frac{A}{a'}=\frac{B}{b} \qquad \cdots \qquad \cdots \quad (1)$$

Now let B retain its value b without undergoing any further change, so that B remains constant, while C changes to c, then A will change from a' to a. Thus the change of A from a' to a is due to the change of C to c only, B remaining constant; and since $A \propto C$, we have by definition

$$\frac{a'}{a}=\frac{C}{c} \qquad \cdots \qquad \cdots \quad (2)$$

Multiplying (1) by (2), we have

$$\frac{A}{a'} \times \frac{a'}{a}=\frac{B}{b} \times \frac{C}{c}$$

t is
$$A=\frac{a}{bc}.BC,$$

where $\frac{a}{bc}$ is a constant, for a, b, c are particular values of A, B, C and therefore are constants.

Hence $A \propto BC$ when both B and C vary.

This theorem may be extended to any number of variables. Thus if there be any number of quantities B, C, D ·········such that another quantity A varies as each one of them when the rest are constant, then A varies as their product when they all vary.

Example 1. *If $x \propto y$, and $x=15$ when $y=5$, find the relation between x and y, and find the value of x when $y=9$.*

Since $x \propto y$, we have $x=my$, m being a constant.

Putting $x=15$ and $y=5$, we have $15=m \times 5 \quad \therefore \quad m=3.$

Thus the required relation is $x=3y$.

$\therefore$ when $y=9$, we have $x=3 \times 9=27$.

Example 2. *If y varies directly as x and inversely as z, and $y=6$, when $x=3$ and $z=5$, find the value of y when $x=7$ and $z=15$.*

By hypothesis, $y=m.\dfrac{x}{z}$, when m is constant.

Putting $y=6$, $x=3$ and $z=5$, we have $6=m\times\frac{3}{5}$. $\therefore$ $m=10$.

Thus the relation between x, y and z is $y=10\dfrac{x}{z}$.

Hence putting $x=7$, and $z=15$, $y=10\times\frac{7}{15}=\frac{14}{3}=4\frac{2}{3}$.

Examples 3. *If y varies as the sum of three quantities, the first of which varies as x^2, the second as x, and the third is constant, and $y=3$, 10, 21, when $x=2$, 3, 4 respectively; find the relation between x and y.*

Since by the question, $y \propto ax^2+bx+c$, where a, b, c are constants, we have

$y=k(ax^2+bx+c)=kax^2+kbx+kc=lx^2+mx+n$,
where l, m, n are written for ka, kb, kc respectively, k being a constant.

Now substituting the three pairs of given values of x and y we have,

$$4l+2m+n=3 \quad \cdots \quad \cdots (1)$$
$$9l+3m+n=10 \quad \cdots \quad \cdots (2)$$
$$16l+4m+n=21 \quad \cdots \quad \cdots (3)$$

Subtracting (1) from (2), $5l+m=7$ ... ... (4)

Subtracting (2) from (3), $7l+m=11$ ... ... (5)

Subtracting (4) from (5), $2l=4$

$\therefore$ $l=2$, and hence $m=-3$, $n=1$.

Thus the relation between x and y is $y=2x^2-3x+1$.

Example 4. *If $x+y$ varies as z when y is constant, and $x+z$ varies as y when z is constant, show that when both y and z vary $x+y+z$ varies as yz.* [*Cal. Int.* 1941]

Since $x+y\propto z$ when y is constant, we have $x+y=mz$.

$\therefore$ $x+y+z=(m+1)z$.

Hence $x+y+z\propto z$, when y is constant ... (1)

Again, since $x+z\propto y$, when z is constant, $x+z=ny$.

$\therefore$ $x+y+z=(n+1)y$.

Hence $x+y+z\propto y$, when z is constant ... (2)

$\therefore$ from (1) and (2), $x+y+z\propto yz$ when both y and z vary.

Example 5. *If $x^2+y^2 \propto xy$, show that $x+y \propto x-y$.*

By the question, we have

$$x^2+y^2=mxy, \text{ where } m \text{ is constant.}$$

$$\therefore \quad \frac{x^2+y^2}{2xy}=\frac{m}{2}.$$

$$\therefore \quad \frac{x^2+y^2+2xy}{x^2+y^2-2xy}=\frac{m+2}{m-2} \text{ [by \textit{Componendo and Dividendo.}]}$$

$$\therefore \quad \frac{x+y}{x-y}=\sqrt{\frac{m+2}{m-2}} \quad \text{[taking the square root]}$$

$$\therefore \quad x+y=\sqrt{\frac{m+2}{m-2}}(x-y).$$

Now $\sqrt{\dfrac{m+2}{m-2}}$ is constant, because m is constant.

Hence $x+y \propto x-y$.

EXERCISE 16.

1. If y varies as x, and $y=7$ when $x=5$, find the value when $x=20$.

2. If y varies as x, and $y=13$ when $x=8$, find the relation between x and y.

3. If y varies inversely as x, and $y=b$ when $x=a$, find the value of y when $x=c$.

4. A varies as B and C jointly; if $A=2$. when $B=\frac{3}{5}$ and $C=\frac{10}{27}$, find C when $A=54$ and $B=3$. [Cal. Int. 1920]

5. If P varies inversely as Q, and $P=7$ when $Q=3$, find P when $Q=2\frac{1}{3}$. [Cal. Int. 1919]

6. If x varies directly as y and inversely as z, and $x=a$, when $y=b$ and $z=c$, find the value of x when $y=b^2$ and $z=c^2$. [Cal. F. A. 1877]

7. If x varies directly as the square of y and inversely as the cube root of z, and if $x=2$ when $y=4$ and $z=8$, find the value of y when $x=3$ and $z=27$. [Cal. Int. 1917]

8. If y is equal to the sum of two quantities one of which varies as x and the other as $\sqrt{x}$, and $y=10$ when $x=4$, and $y=21$ when $x=9$, find the relation between x and y.

9. If y varies as the sum of three quantities the first of which varies as x^2, the second as x, and the third is constant, and $y=8$, 22, 42, when $x=3$, 4, 5 respectively, find the relation between x and y; also find the two values of x which make y vanish.

10. If y is equal to the sum of two quantities one of which varies directly as x and the other inversely as x, and if $y=pa+b$ when $x=a$, and $y=a+pb$ when $x=b$, find the two values of x which make $y=pab+1$.

11. If $A \propto B$ when C is constant, and $A \propto C$ when B is constant, prove that when $B \backsim C$, then will $A \propto B^2$ or C^2.

12. If $A^2+B^2 \propto A^2-B^2$, show that $A \propto B$. [Cal. Int. 1923]

13. If $a \propto bc$ and $b \propto ca$, show that c is a constant.

14. If $a \propto b$, show that $a^2-b^2 \propto ab$. [Bom. 1927]

15. If $ax-by \propto \sqrt{xy}$, prove that $x^2+y^2 \propto xy$. [Bom. P. E. 1896]

16. If $p=3q$ and $ap=bq$, prove that $a \propto b$. [Bom. 1929]

17. If $a \propto b$, prove that $\dfrac{a^3-b^3}{a-b} \propto ab$. [Bom. 1933]

18. If $x+y \propto x-y$, show that

(i) $x^3+y^3 \propto xy(x-y)$; (ii) $xy \propto (x+y)(x-y)$. [Bom. 1928]

19. If $x+y \propto x-y$, show that (i) $x^2-y^2 \propto xy$, and (ii) $ax+by \propto px+qy$, a, b, p, and q being all constants. [Cal. Int. 1936]

20. If $\dfrac{1}{x}-\dfrac{1}{y}$ varies inversely as $x-y$, prove that

$$x^2+y^2 \propto xy.$$

21. If $2x-3y \propto y+5x$, prove that $\dfrac{1}{3x^2}+\dfrac{1}{4y^2} \propto \dfrac{1}{2x^2}-\dfrac{2}{3y^2}$. [Bom. 1932]

22. If $x \propto y$ and $y \propto z$, show that

$$x+y+z \propto (yz)^{\frac{1}{2}}+(zx)^{\frac{1}{2}}+(xy)^{\frac{1}{2}}.$$

23. Given that $x+y \propto z+\frac{1}{z}$, and that $x-y \propto z-\frac{1}{z}$, find the relation between x and z, provided that $z=2$ when $x=3$ and $y=1$. [Bom. P. E. 1888 & Cal. Int. 1912]

24. If a, b, c are variables such that $a \propto mb+nc$ and $b \propto pc+qa$, m, n, p and q being all constants, prove that $a \propto b$.

25. If x, y, z be three variable quantities such that $y+z-x$ is constant and that $(x+y-z)(z+x-y)$ varies as yz, prove that $x+y+z$ varies as yz.

26. If a, b, c are variable and n a constant, such that $a \propto b$ and $b \propto c$, show that $a^n+b^n+c^n \propto ab^{n-1}+bc^{n-1}+ca^{n-1}$.

27. If $a^2+b^2 \propto x^2+y^2$, and $ab \propto xy$, and $\frac{a}{b}+\frac{b}{a}=\frac{x}{y}+\frac{y}{x}$, show that $a+b \propto x+y$.

28. If $a \propto b \propto c$, show that

$$pa+qb+rc \propto p'(ab)^{\frac{1}{2}}+q'(bc)^{\frac{1}{2}}+r'(ca)^{\frac{1}{2}},$$

p, q, r, p', q', r' being all constants.

29. If $a \propto b+c$, $b \propto c+a$, and $c \propto a+b$, find the relation between the constants of variation.

30. If $ax^2+2hxy+by^2 \propto z^2$, and $lx+my \propto z$, show that $x \propto y$.

31. If $z \propto x+y$, $u \propto x-y$, and $x \propto u+z$, show that in general $y \propto x$. [Bom. P. E. 1895]

32. If $(x+y+z)(y+z-x)(z+x-y)(x+y-z) \propto x^2y^2$, then either $x^2+y^2=z^2$, or, $x^2+y^2-z^2 \propto xy$. [Bom. 1898]

33. If $x \propto y$ and $y \propto z$, and if a, b, c, and a', b', c', be two sets of values of x, y, z, show that

$$\frac{a^2+b^2+c^2}{aa'+bb'+cc'}=\frac{aa'+bb'+cc'}{a'^2+b'^2+c'^2}.$$ [Cal. Int. 1922]

59. Problems involving Variation.

We proceed to work out some problems involving variation.

Example 1. *Find how soon* 20 *men will earn Rs.* 30, *if* 3 *men can earn Rs.* 9 *in* 16 *days.* [*Mad. F. A.* 1885]

It is evident that the amount of money earned varies as the number of men when the number of days is constant, and as the number of days when the number of men is constant.

Hence if x, y, z be respectively the amount of money earned in rupees, the number of men and the number of days, we have

$x \propto y$, when z is constant,

and $x \propto z$, when y is constant.

$x \propto yz$, when both y and z vary ;

so that $x = myz$, where m is constant.

It is given that $x = 9$, when $y=3$ and $z=16$.

Hence, $9 = m \times 3 \times 16$. $\therefore$ $m = \frac{3}{16}$

$$x = \tfrac{3}{16} yz. \qquad (1)$$

Now putting $x=30$ and $y=20$ in equation (1) we have

$$30 = \tfrac{3}{16} \times 20 \times z,$$

whence $z=8$, the required number of days.

Example 2. *Three solid spheres of glass whose radii are* 3, 4 *and* 5 *inches respectively, are melted and formed into a single solid sphere ; find the radius of the sphere so formed, given that the volume of a sphere varies as the cube of its radius.*

Let V_1, V_2 and V_3 be the volumes of the three spheres and V the volume of the resulting sphere, then by the question,

$$V = V_1 + V_2 + V_3 \quad \dots \quad \dots \quad (1)$$

Since the volume of sphere $= m \times$ (radius)^3, we have $V_1 = m \times 3^3 = 27m$, $V_2 = m \times 4^3 = 64m$, $V_3 = m \times 5^3 = 125m$, and $V = mr^3$ where r is the radius of the resulting sphere.

Hence from (1), $mr^3 = 27m + 64m + 125m = 216m$,

or, $r^3 = 216$.

$r = 6$ inches.

Example 3. *Supposing that the velocity of a steamer varies inversely as the area of its greatest section when the tonnage is constant and inversely as the tonnage when the area is constant, and that a steamer whose section is* 200 *sq. ft. and tonnage* 1000 *goes* 15 *miles per hour, find the velocity of a steamer whose section is* 250 *sq. ft. and tonnage* 1200. [*Bom. P. E.* 1893]

Let v be the velocity, s the area of the greatest section, and t the tonnage, then, by supposition,

$$v \propto \frac{1}{s}, \text{ when } t \text{ is constant,}$$

and $$v \propto \frac{1}{t}, \text{ when } s \text{ is constant.}$$

$\therefore$ $$v \propto \frac{1}{st}, \text{ when both } s \text{ and } t \text{ vary.}$$

Hence $$v = \frac{m}{st} \text{ or } vst = m, \text{ where } m \text{ is constant,}$$

Putting $v=15$, $s=200$ and $t=1000$, we have

$$m = 15 \times 200 \times 1000$$

$$\therefore \quad vst = 15 \times 200 \times 1000 \quad \ldots \quad \ldots \quad (1)$$

Putting $s=250$, and $t=1200$ in (1), we have

$$v \times 250 \times 1200 = 15 \times 200 \times 1000.$$

$$\therefore \quad v = 10 \text{ miles per hour.}$$

EXERCISE 17.

1. How long will 25 men take to plough 30 acres, if 5 men take 9 days to plough 10 acres of land? [Cal. Int. 1934]

2. If 6 men can earn 21 rupees in 4 days, in how many days will 8 men earn 35 rupees?

3. If 15 men can reap 32 acres of land in 6 days, working 8 hours per day, how many days will 18 men take to reap 40 acres of land working 10 hours per day?

4. Two circular plates of the same thickness and radii 5 and 12 inches respectively are melted and formed into one circular plate of the same thickness. Find the radius, having given that the area of a circle varies as the square of its radius.

5. Two globes of gold have their radii equal to r and r'; they are melted and formed into a single globe. Find its radius. (The volume of a globe varies as the cube of the radius.) [Cal. Int. 1931]

6. It is known that the volume of a sphere varies as the cube of its radius ; show that the sum of three spheres of radii 1, 6 and 8 inches respectively is equal to the volume of a sphere of radius 9 inches.

7. The time of a complete oscillation of a pendulum varies as the square root of its length. If a pendulum of length 44 inches makes three complete oscillations in 11 seconds, find the time of a complete oscillation of a pendulum of length 35·64 inches.

8. The time of oscillation of a pendulum varies as the square root of its length. If a pendulum of length 40 inches oscillates once in a second, what is the length of the pendulum oscillating once in 2·5 seconds ? [Cal. Int. 1913]

9. The length of a pendulum which vibrates seconds is 39·14 inches, and the length varies as the square of the number of seconds in a vibration. What would be the vibration of a monster pendulum 40 ft. long ? [All. I. E. 1895]

10. If m sovereigns in a row stretch as far as n pennies, and p sovereigns in a pile are as high as q pennies, compare the value of equal bulks of gold and copper, assuming that the area of a circle varies as the square of its radius.
[Bom. P. E. 1891]

11. The value of a silver coin varies directly as the square of its diameter while its thickness remains the same and directly as its thickness while its diameter remains the same. Two silver coins have their diameters in the ratio of 4 : 3 ; find the ratio of their thicknesses if the value of the first is four times the value of the second. [Bom. P.E. 1885]

12. Pressure in a liquid varies as the depth when the density is constant, and varies as the density when the depth is constant. The pressure is 1 when the depth is 32 and the density is 1. Find the depth at which the pressure is 2 when the density is 16. [Cal. Int. 1921]

13. The volume of a circular cylinder varies as the square of the radius of the base when the height is the same, and as the height when the base is the same. The volume is 88 cubic feet when the height is 7 ft. and the radius of the base is 2 ft. ; what will be the height of a cylinder on a base of radius 9 ft. when the volume is 396 cubic feet ? [Bom. 1926]

14. It is given that the weight of a body above ground varies inversely as the square of its distance from the centre of the earth, and that within the earth the weight varies as the distance from the centre. The weight of a body on

the surface of the earth is 200 lbs.; what will be its weight (i) at a height of 1000 miles; and (ii) at a depth of 3 miles within the earth, the radius of the earth being supposed to be 4000 miles?

15. The weight of a sphere varies directly as the cube of its radius and as the density of the material of which it is made. The radii of two spheres are as 16 : 7, and the densities of their materials as 2 : 3. If the weight of the first is 256 lbs., find the weight of the second. [Bom. 1930]

16. The volume of a pyramid varies as the area of its base when its altitude is constant and varies as its altitude when the area of the base is constant. When the area of the base of a pyramid is 16 sq. ft. and its cubical contents are 32 cub. ft., its height is 6 ft. Find the area of the base of another pyramid whose height is 20 ft. and cubical contents 120 cub. ft. [Bom. 1931]

17. The attendance at a Professor's lectures varies directly as the Professor's power of exposition and inversely as the square of the number of lectures delivered. If 64 students attend the lectures of Professor *A* who delivers a course of 12 lectures, find the number of students who attend the lectures of Professor *B* who delivers a course of 16 lectures and who possesses twice as much power of exposition as Professor *A*. [Bom. P. E. 1875]

18. The pressure of the wind on a plane surface varies jointly as the area of the surface and the square of the wind velocity. The pressure on a square foot is 1 lb. when the wind is moving at the rate of 15 miles per hour; find the velocity of the wind when the pressure on a square yard is 16 lbs. [Bom. 1925]

19. The consumption of coal by a locomotive varies as the square of its velocity; when the speed is 16 miles an hour, the consumption of coal per hour is 2 tons. If the price of coal be 10*s.* per ton and the other expenses of the engine be 11*s.* 3*d.* an hour, find the least cost of a journey of 100 miles.

20. The expenses of a hostel are partly constant and partly vary as the number of inmates. The expenses were Rs. 2000 when the inmates were 120, and Rs. 1700 when the inmates were 100. Find the number of inmates when the expenses were Rs. 1880. [Bom. 1927]

CHAPTER VII

QUADRATIC EQUATIONS.

60. Definitions.

An equation which involves the square of the unknown quantity but no higher power is called an *equation of the second degree*, or a *quadratic equation*.

Quadratic equations are of two kinds, *viz.*, Pure Quadratics and Adfected Quadratics. *Pure Quadratics* are those in which, besides the constant term, only the second power of the unknown quantity occurs and not the first. *Adfected Quadratics* are those in which the first as well as the second power of the unknown quantity occur.

Thus, $5x^2=80$ is a pure quadratic equation; and $2x^2-7x+5=0$, is an adfected quadratic equation.

The general form of a pure quadratic equation is $ax^2+b=0$, and that of an adfected quadratic equation is $ax^2+bx+c=0$, and all quadratic equations are ultimately reducible to one or other of these forms.

1. Pure Quadratic Equations.

61. Methods of Solution.

A pure quadratic equation may be solved by either of two methods, *viz.*, (*i*) by transposing all the terms to one side, and then *factorizing*; and (*ii*) by transposing the terms involving x to one side and the constant terms on the other, and then *taking the square root.*

For instance, let us solve the pure quadratic equation

$$x^2=a^2 \qquad \cdots \qquad \cdots \qquad (1)$$

(*i*) *Method of Factorization.*

Transposing, $\qquad x^2-a^2=0$

or, $\qquad (x-a)(x+a)=0$

$\therefore \qquad$ either $x-a=0$, or $x+a=0$.

Hence, $\qquad x=a$ or $-a$.

(*ii*) *Method of square roots.*

From (1), taking the square root, $\pm x = \pm a$,

i.e., $x = a,\ x = -a,\ -x = a,\ -x = -a.$

It is clear that the third solution is the same as the second, and the fourth the same as the first.

Thus there are only two solutions, and they are $x = \pm a$.

It appears therefore that it is unnecessary to attach the double sign $\pm$ to both sides, it is sufficient to attach it to one side ; *e.g.*, $x = \pm a$.

62. Number and Nature of the Roots.

Since every quantity has two square roots equal in magnitude but opposite in sign, it follows that *a pure quadratic equation has always two roots equal in magnitude but opposite in sign.* [See Art. 10.]

The roots, however, may be either real or *imaginary* ; if the co-efficient of x^2 be positive, the roots are real when the constant term is negative as in the equation $x^2 - a^2 = 0$; and imaginary, when the constant term is positive, as in the equation $x^2 + a^2 = 0$.

For instance, let us solve $x^2 + a^2 = 0$.

Transposing, $x^2 = -a^2 = a^2(-1)$.

Taking the square root, $x = \pm a\sqrt{-1}$, which are imaginary quantities. [See Art. 8.]

When the constant term is zero, the roots are both equal to zero, as in the equation $x^2 = 0$.

Example 1. *Solve* $4(3x^2 - 2) - 2(4x^2 - 3) = 3(2 + x)(2 - x)$.

We have $12x^2 - 8 - 8x^2 + 6 = 3(4 - x^2)$

or, $4x^2 - 2 = 12 - 3x^2$

or, $7x^2 = 14$

or, $x^2 = 2$

$\therefore$ $x = \pm\sqrt{2}$.

Example 2. *Solve* $\sqrt{a + x} + \sqrt{a - x} = 2$.

Squaring, $2a + 2\sqrt{a^2 - x^2} = 4$,

or, $\sqrt{a^2 - x^2} = 2 - a$ [transposing and dividing by 2]

Squaring again, $a^2 - x^2 = 4 - 4a + a^2$

$\therefore$ $x^2 = 4a - 4$

$\therefore$ $x = \pm 2\sqrt{a - 1}$.

Example 3. *Solve* $\dfrac{\sqrt{(x^2+1)}+\sqrt{(x^2-1)}}{\sqrt{(x^2+1)}-\sqrt{(x^2-1)}}=a.$

By *Componendo and Dividendo*, we have

$$\frac{\sqrt{(x^2+1)}}{\sqrt{(x^2-1)}}=\frac{a+1}{a-1}$$

Squaring, $$\frac{x^2+1}{x^2-1}=\left(\frac{a+1}{a-1}\right)^2=\frac{a^2+2a+1}{a^2-2a+1}.$$

Again, by *Componendo and Dividendo*,

$$x^2=\frac{a^2+1}{2a}.$$

$$\therefore \qquad x=\pm\sqrt{\left(\frac{a^2+1}{2a}\right)}.$$

EXERCISE 18.

Solve the following equations :

1. $\frac{3}{4}x^2+2=\frac{1}{4}x^2+3.$

2. $\frac{1}{3}(x-3)^2+\frac{1}{4}(x+4)^2=28.$

3. $\dfrac{7x}{4}-\dfrac{20}{x}=\dfrac{x}{2}.$

4. $\dfrac{b}{a-bx}+\dfrac{a}{b+ax}=\dfrac{a^2+b^2}{(a^2-b^2)x}.$

5. $\dfrac{1}{a+x}+\dfrac{1}{a-x}=1.$

6. $\dfrac{x+2}{x-2}+\dfrac{x+3}{x-3}=2\left(\dfrac{5x+12}{5x+11}\right).$

7. $\dfrac{x+5}{x-5}+\dfrac{x-5}{x+5}=\dfrac{17}{4}.$

8. $\dfrac{5}{x^2-2}+\dfrac{9}{3x^2-2}=\dfrac{8}{x^2+1}.$

9. $\dfrac{x}{1-\dfrac{x}{x+2}}=x+4.$

10. $\dfrac{x+a+b}{x+a-b}=\dfrac{b+a-x}{b-a+x}.$

11. $\dfrac{ax+b}{cx+d}=\dfrac{a+bx}{c+dx}.$

12. $\dfrac{x^2-1}{x^2-5}+\dfrac{x^2-5}{x^2-9}+\dfrac{x^2-9}{x^2-1}=3.$

13. $\dfrac{x}{a}+\dfrac{a}{x}=\dfrac{x}{b}+\dfrac{b}{x}.$

14. $\frac{1}{2}(x^2-3)+\frac{1}{3}(x^2-4)+\frac{1}{6}(x^2-7)=0.$

15. $\dfrac{x^2-1}{x^2-4}-\dfrac{x^2-5}{x^2-8}=\dfrac{x^2-2}{x^2-5}-\dfrac{x^2-6}{x^2-9}.$

16. $\sqrt{(2x+3)}+\sqrt{(2x-3)}=\sqrt{6x}.$

17. $x^2+3=\sqrt{(x^2+1)(x^2+6)}.$

18. $\sqrt{(x^2+1)}-\sqrt{(x^2-1)}=1.$

19. $\sqrt{(x^2+9)}+\sqrt{(x^2-9)}=4+\sqrt{34}.$ [Cal. 1927]

20. $\dfrac{2a\sqrt{(1+x^2)}}{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}=a+b.$ 21. $\dfrac{2ax}{x+\sqrt{(x^2+1)}}=b.$

22. $\dfrac{\sqrt{1+x}-1}{\sqrt{1-x}+1}+\dfrac{\sqrt{1-x}+1}{\sqrt{1+x}-1}=2a.$

23. $\sqrt{\dfrac{x+a}{x-a}}+\sqrt{\dfrac{x-a}{x+a}}=2b.$

24. $\dfrac{1}{\sqrt{(1-x)}+1}+\dfrac{1}{\sqrt{(1+x)}-1}=\dfrac{1}{x}.$ 25. $\dfrac{a+\sqrt{(a^2-x^2)}}{a-\sqrt{(a^2-x^2)}}=4.$

II. Adfected Quadratic Equations.

63. Methods of Solution.

As in the case of pure quadratics, so in the solution of adfected quadratic equations, there are really only two methods of solution, *viz.*, (*i*) transposing all the terms to one side, and then *factorizing* ; and (*ii*) transposing all the terms involving x to one side and the constant terms to the other and by the introduction of suitable terms and factors to the side involving x making it a *perfect square*, and then *taking the square root*.

These two methods are called the methods of *factorization* and of *completing the square* respectively.

Completing the square, again, can be done in two ways : one of these is adopted in the ordinary method and the other in what is called the Hindu method ascribed to Shreedharacharyya.

64. First method : Factorization.

Transpose all the terms to one side and factorize, and equate each of the factors to zero. When the factorization is easily done, this is the shortest and simplest method.

Example 1. *Solve* $3x^2-13x+14=0$.

Factorizing, $3x^2-13x+14=(x-2)(3x-7)$

$\therefore$ $(x-2)(3x-7)=0$

$\therefore$ either $x-2=0$, or, $3x-7=0$

$\therefore$ $x=2$ or $\frac{7}{3}$.

Example 2. *Solve* $(a-b)x^2-(a+b)x+2b=0.$

The left-hand side $=(a-b)x^2-(a-b)x-2bx+2b$

$$=(a-b)x(x-1)-2b(x-1)$$
$$=(x-1)\{(a-b)x-2b\}$$

$\therefore$ $(x-1)\{(a-b)x-2b\}=0$

$\therefore$ either $x-1=0$, or, $(a-b)x-2b=0.$

$$x=1 \text{ or } \frac{2b}{a-b}.$$

EXERCISE 19.

Solve the following equations

1. $2x^2-9x+7=3.$
2. $3x^2+4x-15=0.$
3. $14x^2+29x-15=0.$
4. $27x^2+12x+1=0.$
5. $4(x+2)(x-2)+(x+1)^2=1.$
6. $7x(3x+1)=2(4x+5).$
7. $x^2+(a+b)x+(a+c)(b-c)=0.$
8. $a(x^2+1)=x(a^2+1).$
9. $x^2+(a+b-2c)x+(a-c)(b-c)=2(a-b)^2.$
10. $(a+b)x^2+(a-b)x=\dfrac{ab}{a+b}.$
11. $(a+b)x^2+cx=a+b+c.$
12. $(x-4a)^2+(x-a)(x-7a)+7a^2=0.$
13. $x^2-2(a+b)x-ab(a-2)(b+2)=0.$
14. $ab\{(x-a)^2+(x-b)^2\}=(a^2+b^2)(x-a)(x-b).$
15. $abx^2-(a^3+b^3)x+a^2b^2=0.$

65. Second Method : Completing the Square.

In this article we give the ordinary method of completing the square. The Hindu method of completing the square will be given in the next article.

Let us take adfected quadratic equation in its general form $$ax^2+bx+c=0.$$

By transposition, $ax^2+bx=-c.$

Dividing both sides by a, $x^2+\dfrac{b}{a}x=-\dfrac{c}{a}.$

Adding $\left(\frac{b}{2a}\right)^2$, *i.e.*, the square of half the co-efficient of x, to both sides, we have

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}$$

or
$$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$$

$\therefore$ taking the square root $x+\frac{b}{2a}=\frac{\pm\sqrt{b^2-4ac}}{2a}$

$\therefore$
$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \qquad \ldots \text{(A)}$$

Thus the roots of the quadratic equation $ax^2+bx+c=0$ are $\frac{-b+\sqrt{b^2-4ac}}{2a}$ and $\frac{-b-\sqrt{b^2-4ac}}{2a}$.

The general expression (A) for the roots of a quadratic is very important. Particular equations in any given case may be solved either by following step by step this method or by assuming the general expression just obtained.

Cor. 1. If the equation be of the form $ax^2+2bx+c=0$, the roots are $\frac{-b\pm\sqrt{b^2-ac}}{a}$.

Cor. 2. If the equation be of the from $x^2+px+q=0$, the roots are $\frac{-p\pm\sqrt{p^2-4q}}{2}$.

86. Number and Nature of Roots.

It appears from the above that *an adfected quadratic equation has two roots.* [See Art. 10.]

The nature of the roots evidently depends upon the value of the expression under the radical sign, b^2-4ac. Thus :

(*i*) If b^2-4ac is *positive*, *i.e.*, if $b^2>4ac$, then the roots are *real and unequal*; and particularly, if b^2-4ac is a *perfect square*, the roots are *rational* (if a, b, c are rational); otherwise, the roots are irrational.

(*ii*) If b^2-4ac is *zero*, *i. e.*, if $b^2=4ac$, then the roots are *real and equal, and also rational* (if a, b, c are rational). In this case, the expression ax^2+bx+c becomes a *perfect square*. [See Art. 25, Ex.1.]

(*iii*) If b^2-4ac is *negative*, *i. e.*, if $b^2<4ac$, then the roots are *imaginary and unequal.*

Example 1. *Solve* $6x^2-x-12=0$.

(*i*) *By completing the square* :

By transposition, $\quad 6x^2-x=12$

Dividing by 6, $\quad x^2-\frac{1}{6}x=2$

Adding $(-\frac{1}{12})^2$, $\quad x^2-\frac{1}{6}x+(-\frac{1}{12})^2=2+\frac{1}{144}=\frac{289}{144}$

or, $\quad (x-\frac{1}{12})^2=(\frac{17}{12})^2$

$\therefore \quad x-\frac{1}{12}=\pm\frac{17}{12}$

$\therefore \quad x=\frac{1}{12}\pm\frac{17}{12}=\frac{3}{2}$ or $-\frac{4}{3}$.

(*ii*) *By using the formula* :

In this equation, $a=6$, $b=-1$, $c=-12$;

$$\therefore \quad x=\frac{-(-1)\pm\sqrt{(-1)^2-4.6.(-12)}}{2\times 6}$$

$$=\frac{1\pm\sqrt{1+288}}{12}=\frac{1\pm 17}{12}$$

$$=\frac{3}{2} \text{ or } -\frac{4}{3}.$$

Thus the roots are $\frac{3}{2}$ and $-\frac{4}{3}$.

Example 2. *Solve* $x^2+x+1=0$.

Here $a=1$, $b=1$, $c=1$. Hence by the formula

$$x=\frac{-1\pm\sqrt{(1)^2-4.1.1}}{2.1}=\frac{-1\pm\sqrt{-3}}{2},$$

so that the roots are imaginary.

Example 3. *Solve* $(a-b)x^2-(c-a+b)x-c=0$.

By using the formula, we have

$$x=\frac{(c-a+b)\pm\sqrt{(c-a+b)^2+4c(a-b)}}{2(a-b)}.$$

Now the expression under the radical sign

$$=(c-a+b)^2+4c(a-b)$$

$$=\{c-(a-b)\}^2+4c(a-b)$$

$$=c^2-2c(a-b)+(a-b)^2+4c(a-b)$$

$$=c^2+2c(a-b)+(a-b)^2$$
$$=(c+a-b)^2.$$
$$x=\frac{(c-a+b)\pm(c+a-b)}{2(a-b)}$$
$$\frac{2c}{2(a-b)} \text{ or } \frac{-2a+2b}{2(a-b)}$$
$$\frac{c}{a-b} \text{ or } -1$$

EXERCISE 20.

Solve the following equations

1. $6x^2-x-2=0.$
2. $24x^2-2x-35=0.$
3. $3x^2-5x+7=0.$
4. $4(5x^2+1)=3(x+2).$
5. $3x(1-x)=x+1.$
6. $15x^2-44x+32=0.$
7. $13x^2+5=2(x+8).$
8. $x^2-x+1=0.$
9. $(a+b)x^2-2ax+(a-b)=0.$
10. $x^2+2ab=b^2+2ax.$
11. $(a^2-b^2)x^2-ab=(a^2+b^2)x.$
12. $(a-x)^2+(b-x)^2=(a-b)^2.$
13. $(a-b)x^2+(c-a)x+(b-c)=0.$
14. $ab(x^2+1)=(a^2+b^2)x.$
15. $(a^2-b^2)x^2+2cax+c^2=0.$
16. $(a^2-b^2)(x^2-1)=4abx.$
17. $a(b-c)x^2+b(c-a)x+c(a-b)=0.$
18. $(c+a-2b)x^2+(a+b-2c)x+(b+c-2a)=0.$

67. The Hindu Method.

Let the quadratic equation be $ax^2+bx+c=0.$

Transposing, we have $\quad ax^2+bx=-c.$

Multiplying by $4a$, $\quad 4a^2x^2+4abx=-4ac$.

Adding b^2 to each side, $4a^2x^2+4abx+b^2=b^2-4ac$

or, $\quad (2ax+b)^2=b^2-4ac$

Taking the square root, $\quad 2ax+b=\pm\sqrt{b^2-4ac},$

or, $\quad 2ax=-b\pm\sqrt{b^2-4ac}$;

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

This is the process laid down by the Hindu mathematician Shreedharacharyya, as quoted by Bhaskaracharyya in the fifth chapter of his *Vijaganita* ; *viz.*

चतुराहतवर्गसमैः रूपैः पक्षद्वयं गुणयेत्।
अव्यक्तवर्गरूपैर्युक्तौ पक्षौ ततो मूलम्॥

Example 1. *Solve* $2x^2-5x+3=0$.

By transposition, $2x^2-5x=-3$

Multiplying both sides by 4×2

$$16x^2-40x=-24$$

Adding 5^2 *i.e.*, 25 to both sides,

$$16x^2-40x+25=25-24$$

or, $$(4x-5)^2=1$$

$\therefore$ $$4x-5=\pm 1$$

$\therefore$ $$x=\frac{5\pm 1}{4}=\tfrac{3}{2} \text{ or } 1.$$

EXERCISE 21.

Solve the following equations :

1. $3x^2-7x+2=0$.
2. $14x^2-29x+12=0$.
3. $12x^2=x+1$.
4. $7x^2+4ax=3a^2$.
5. $(2x-1)(3x+2)+(6x-1)=0$.
6. $9x^2-6x=7$.
7. $(x-1)(6x-5)=2$.
8. $2bx^2-abx=3(2x-a)$.
9. $(a-2)x^2+(b+2)x=a+b$.
10. $bcx^2+ab=(b^2+ca)x$.
11. $(a+1)x^2+2ax+(a-1)=0$.
12. $abx^2-(a+b)x+1=0$.

III. Equations reducible to Quadratics.

68. Method of Substitution.

Many equations which are not quadratics may be solved like quadratic equations. The method in practically all such cases is to make use of a suitable *substitution* which reduces the given equation to the quadratic form.

Hence equations which are reducible to a quadratic may be written in the general form

$$Ay^2+By+C=0,$$

where y is some function of x.

Example 1. *Solve :* $x^4-13x^2+36=0$.

Putting $x^2=z$, the equation becomes

$$z^2-13z+36=0$$

or, $(z-4)(z-9)=0$

$\therefore$ $z=4$ or 9

i.e., $x^2=4$ or 9

$\therefore$ $x=\pm 2$ or ± 3.

Hence the roots are ± 2 and ± 3.

Example 2. *Solve :* $ab(x^{\frac{1}{n}}+x^{-\frac{1}{n}})=a^2+b^2$.

Multiplying by $x^{\frac{1}{n}}$, $ab(x^{\frac{2}{n}}+1)=(a^2+b^2)x^{\frac{1}{n}}$

or, $abx^{\frac{2}{n}}-(a^2+b^2)x^{\frac{1}{n}}+ab=0$

Putting $x^{\frac{1}{n}}=z$, the equation becomes

$$abz^2-(a^2+b^2)z+ab=0$$

or, $(bz-a)(az-b)=0$

$$z=\frac{a}{b} \text{ or } \frac{b}{a}$$

i.e., $x^{\frac{1}{n}}=\frac{a}{b}$ or $\frac{b}{a}$

$$x=\frac{a^n}{b^n} \text{ or } \frac{b}{a}$$

Example 3. *Solve :* $(x+1)(x+2)(x+3)(x+4)=8$.

Re-arranging the factors on the left-hand side, we have

$$(x+1)(x+4)(x+2)(x+3)=8$$

or, $(x^2+5x+4)(x^2+5x+6)=8$.

Putting $x^2+5x=z$, the equation becomes,

$$(z+4)(z+6)=8$$

or, $z^2+10z+16=0$

or, $(z+2)(z+8)=0$

$\therefore$ $z=-2$ or -8.

When $z=-2$, we have

$$x^2+5x=-2$$

or, $x^2+5x+2=0$

$$x=\frac{-5\pm\sqrt{25-8}}{2}=\frac{-5\pm\sqrt{17}}{2}$$

When $z=-8$, we have

$$x^2+5x=-8$$

or, $x^2+5x+8=0$

$$x=\frac{-5\pm\sqrt{25-32}}{2}=\frac{-5\pm\sqrt{-7}}{2}$$

Hence, the reqd. roots are $\frac{1}{2}(-5\pm\sqrt{17})$ and $\frac{1}{2}(-5\pm\sqrt{-7})$.

Example 4. *Solve :* $x^4+x^3-4x^2+x+1=0.$

The left-hand side is a *reciprocal expression*, *i.e.*, an expression which remains unchanged when for the variable is substituted its reciprocal; in such an expression, the coefficient of terms equidistant from the beginning and the end are numerically equal. Such an equation is called a *reciprocal equation.* The method of solution is shown below.

Dividing by x^2, we have $x^2+x-4+\frac{1}{x}+\frac{1}{x^2}=0.$

Re-arranging, $\left(x^2+\frac{1}{x^2}\right)+\left(x+\frac{1}{x}\right)-4=0$

or, $\left(x+\frac{1}{x}\right)^2+\left(x+\frac{1}{x}\right)-6=0.$

Putting $x+\frac{1}{x}=z$, the equation becomes

$$z^2+z-6=0$$

or, $(z-2)(z+3)=0$

$\therefore$ $z=2$, or -3.

Taking the first value,

$$x+\frac{1}{x}=2$$

or, $x^2-2x+1=0$

or, $(x-1)^2=0$

$\therefore$ $x=1, 1.$

Taking the second value,

$$x+\frac{1}{x}=-3$$

or, $x^2+3x+1=0$

$\therefore$ $x=\frac{-3\pm\sqrt{9-4}}{2}=\frac{-3\pm\sqrt{5}}{2}.$

EXERCISE 22.

Solve the following equations :

1. $x^4-14x^2+40=0.$
2. $x^{10}+31x^5=32.$
3. $\sqrt[4]{x}+7\sqrt{x}=30.$
4. $x^{2n}-2ax^n=b^2.$
5. $x^4-3x^3-2x^2-3x+1=0.$
6. $x^3+40=18x\sqrt{x}.$ [Cal. F.A. 1882]
7. $4x^4-16x^3+23x^2-16x+4=0.$
8. $(x-1)(x-2(x-3)=24.$
9. $x+x^{-2}=a+a^{-2}.$ [Cal. F.A. 1895]
10. $(x+a)(x+b)(x+c)=abc.$
11. $(x+1)^5=16(x^5+1).$
12. $x^4+(x+1)^4=17.$

13. $(x+a)^3+(x+b)^3=8x^3+(a+b)^3.$

14. $x(x+1)(x+2)(x+3)=24.$

15. $(x+2)(x+4)(x+6)(x+8)=105.$

16. $(x+2)(3x+1)(x-1)(3x+2)=224.$ [Bom. P.E. 1892]

17. $(2x-1)^2(4x-1)(4x-3)=2.$

18. $2(x^2-3x+1)^2+5(x^2-3x+1)+3=0.$ [Cal. F.A. 1873]

19. $x^2+\sqrt{x^2-5}=11.$ [Pun. I.E. 1893]

20. $3x+2\sqrt{x^2-3x+9}=x^2+6.$ [Cal. F.A. 1886]

21. $x(x+1)+3\sqrt{2x^2+6x+5}=2(12-x)+1.$

22. $(x+6)^2+2\sqrt{x}.\,(x+6)=138+\sqrt{x}.$ [Cal. F.A. 1863]

23. $(a-x)^{\frac{1}{4}}+(b-x)^{\frac{1}{4}}=(a+b-2x)^{\frac{1}{4}}.$

24. $\dfrac{x}{x^2+x+5}+\dfrac{5}{\sqrt{x^2+x+5}}=\dfrac{116}{25x}.$

25. $(1+x)^{\frac{2}{5}}+(1-x)^{\frac{2}{5}}=3\,(1-x^2)^{\frac{1}{5}}.$

26. $\sqrt[m]{(1+x)^2}-\sqrt[m]{(1-x)^2}=\sqrt[m]{1-x^2}.$

27. $(x+4)(x+1)-\sqrt{(x+5)(x-3)}=3x+31.$ [Cal. F.A. 1877]

28. $(x+m)^{\frac{2}{5}}+(x-m)^{\frac{2}{5}}=\left(n+\dfrac{1}{n}\right)\left(x^2-m^2\right)^{\frac{1}{5}}.$ [Mad. F.A. 1884]

29. $x^{\frac{5}{12}}-\frac{1}{2}.\dfrac{p^2-q^2}{p^2+q^2}\left(x^{\frac{1}{2}}+x^{\frac{1}{3}}\right)=0.$ [Mad. F.A. 1882]

30. $\dfrac{x^2-a^2-b^2}{c^2}+\dfrac{c^2}{x^2-a^2-b^2}=2.$ [Cal. F.A. 1883]

31. $\dfrac{(x-2)^2}{x^2-4x}+\dfrac{2}{(x-2)^2}=4.$ [Cal. F.A. 1889]

32. $\dfrac{x-\sqrt{2x}+1}{x+\sqrt{2x}+1}=\dfrac{a}{b}.$ [Cal. F.A. 1862]

CHAPTER VIII

THEORY OF QUADRATIC EQUATIONS AND QUADRATIC EXPRESSIONS.

69. A quadratic equation cannot have more than two roots. (9, 10, 11, 25, 33, 50)

Let us take the quadratic equation in its general form

$$ax^2+bx+c=0.$$

Now, $\quad ax^2+bx+c=a\left(x^2+\dfrac{b}{a}x+\dfrac{c}{a}\right)$

$$=a\left\{\left(x+\frac{b}{2a}\right)^2-\frac{b^2-4ac}{4a^2}\right\}$$

$$=a\left\{\left(x+\frac{b}{2a}\right)-\frac{\sqrt{b^2-4ac}}{2a}\right\}\left\{\left(x+\frac{b}{2a}\right)+\frac{\sqrt{b^2-4ac}}{2a}\right\}$$

$$=a\left(x-\frac{-b+\sqrt{b^2-4ac}}{2a}\right)\left(x-\frac{-b-\sqrt{b^2-4ac}}{2a}\right).$$

Hence the only two values of x which can make ax^2+bx+c vanish are

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

No other value of x can make any of the factors vanish and hence can be a root of the given equation.

Thus a quadratic equation has always two roots but cannot have more than two roots.

Indirect Proof:

This theorem can also be proved by the *indirect method* as follows:

Let us take the general quadratic equation $ax^2+bx+c=0$, and if possible, let this equation have three different roots α, β and γ.

Since each of these roots satisfies the equation, we must have

$$a\alpha^2+b\alpha+c=0 \quad \dots \quad \dots \quad (1)$$
$$a\beta^2+b\beta+c=0 \quad \dots \quad \dots \quad (2)$$
$$a\gamma^2+b\gamma+c=0 \quad \dots \quad \dots \quad (3)$$

Subtracting (2) from (1), we have

$$a(\alpha^2-\beta^2)+b(\alpha-\beta)=0.$$

Since by supposition, $\alpha-\beta$ is not equal to zero, we have dividing by $\alpha-\beta$,

$$a(\alpha+\beta)+b=0 \qquad \cdots \qquad \cdots \quad (4)$$

Similarly, subtracting (3) from (1) and dividing by $\alpha-\gamma$,

$$a(\alpha+\gamma)+b=0 \qquad \cdots \qquad \cdots \quad (5)$$

Again, subtracting (5) from (4),

$$a(\beta-\gamma)=0 \qquad \cdots \qquad \cdots \quad (6)$$

Hence either $a=0$ or $\beta-\gamma=0$, which is impossible ; for a, being the co-efficient of x^2, is not zero and also by hypothesis $\beta-\gamma$ is not zero. Hence our supposition is wrong, and therefore a quadratic equation cannot have more than two roots.

If it happens however that a quadratic equation is satisfied for more than two values of the variable, say α, β and γ, then from (6), since $\beta-\gamma$ is not zero, we must have $a=0$; and since $a=0$, from (4) we must have $b=0$; and again since $a=0$, $b=0$, from (1) we must have $c=0$. Hence the quadratic equation becomes $0.x^2+0.x+0=0$, which is evidently an identity and satisfied for all values of the variable. In other words, *if a quadratic equation has more than two roots it is no longer an equation but an identity.* [See Art. 10.]

Example 1. *Prove that*

$$a^2\frac{(x-b)(x-c)}{(a-b)(a-c)}+b^2\frac{(x-c)(x-a)}{(b-c)(b-a)}+c^2\frac{(x-a)(x-b)}{(c-a)(c-b)}=x^2$$

is an identity. [*Cal. F.A.* 1884]

The equation is a quadratic in x and is obviously satisfied when $x=a$, or $x=b$, or $x=c$. Thus the quadratic is satisfied by three different values of x, and therefore by all values of x ; hence it is an identity.

Example 2. *If* $(x-a)^3+(x-b)^3+(x-c)^3=3(x-d)^3$,

$$(y-a)^3+(y-b)^3+(y-c)^3=3(y-d)^3,$$
$$(z-a)^3+(z-b)^3+(z-c)^3=3(z-d)^3,$$

then will $(w-a)^3+(w-b)^3+(w-c)^3=3(w-d)^3$,

whatever value w may have.

Let us consider the equation

$$(X-a)^3+(X-b)^3+(X-c)^3=3(X-d)^3 \qquad \cdots \quad \cdots \quad (1)$$

which is obviously a quadratic in X. On account of the three

given relations, this quadratic equation in X is evidently satisfied when $X=x$, $X=y$, or $X=z$, that is, for three different values of X; hence it must be an identity and satisfied for any value of X. Hence writing $X=w$ in (1) we have

$$(w-a)^3+(w-b)^3+(w-c)^2=3(w-d)^3.$$

EXERCISE 23.

1. Prove that each of the following is an identity :

(i) $a\dfrac{(x-b)(x-c)}{(a-b)(a-c)}+b\dfrac{(x-c)(x-a)}{(b-c)(b-a)}+c\dfrac{(x-a)(x-b)}{(c-a)(c-b)}=x.$

(ii) $(x+a+b)^3=x^3+a^3+b^3+3(a+b)(x+a)(x+b).$

(iii) $(x-a)^3+(b-x)^3+(a-b)^3=3(a-b)(x-a)(b-x).$

(iv) $(x-a)^2(b-c)+(x-b)^2(c-a)+(x-c)^2(a-b)$
$\qquad\qquad+(b-c)(c-a)(a-b)=0.$

2. If $(a-m)^2x+(a-n)^2y=(a-p)^2z,$
$(b-m)^2x+(b-n)^2y=(b-p)^2z,$
$(c-m)^2x+(c-n)^2y=(c-p)^2z,$
then will $(k-m)^2x+(k-n)^2y=(k-p)^2z$, for all values of k.

3. If $l(x-a)^3+m(x-b)^3+n(x-c)^3=(l+m+n)(x-d)^3,$
$l(y-a)^3+m(y-b)^3+n(y-c)^3=(l+m+n)(y-d)^3,$
$l(z-a)^3+m(z-b)^3+n(z-c)^3=(l+m+n)(z-d)^3,$
then will $l(w-a)^3+m(w-b)^3+n(w-c)^3=(l+m+n)(w-d)^3,$ whatever value w may have.

4. Determine whether each of the following is an equation or an identity :

(i) $2\{(x-a)(x-b)+(a-x)(a-b)+(b-x)(b-a)\}$
$\qquad\qquad=(a-b)^2+(x-a)^2+(x-b)^2.$

(ii) $\dfrac{x-a}{x+a}-\dfrac{x-2a}{x+2a}=\dfrac{x-3a}{x+3a}-\dfrac{x-4a}{x+4a}.$

(iii) $a^2\dfrac{(x-b)(x-c)}{(a-b)(a-c)}+b^2\dfrac{(x-c)(x-a)}{(b-c)(b-a)}=x^2.$

5. If $f(x)=\dfrac{(x-p)(x-q)(x-r)}{x}$, prove the identity

$$\frac{f(a)}{(a-b)(a-c)}+\frac{f(b)}{(b-c)(b-a)}+\frac{f(c)}{(c-a)(c-b)}=1-\frac{pqr}{abc}.$$

70. Discussion of the Roots of a Quadratic.

We have already seen in Art 66, that since the roots of the quadratic equation $ax^2+bx+c=0$ are

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

the nature of the roots evidently depends upon the value of the expression under the radical sign, *viz.*

$$b^2-4ac,$$

which expression is accordingly called the *discriminant* of the equation, as it enables us to discriminate the nature of the roots. Thus :

(*i*) If b^2-4ac is *positive*, *i e.*, if $b^2>4ac$, then the roots are *real and unequal*. And particularly if b^2-4ac is a *perfect square*, the roots are *rational* (if a, b, c are rational), otherwise the roots are irrational.

(*ii*) If b^2-4ac is *zero*, *i.e.*, if $b^2=4ac$, the roots are *real and equal*, and also *rational* (if a, b, c are rational).

(*iii*) If b^2-4ac is negative, *i.e.*, if $b^2<4ac$, the roots are *imaginary and unequal.*

Apart from the value of the discriminant affecting the nature of the roots of the equation, it is interesting to investigate what the roots become if one or more of the co-efficients vanish. Thus :

(*a*) If $c=0$, the equation becomes

$$ax^2+bx=0$$

or $$x(ax+b)=0$$

$\therefore$ $$x=0, \text{ or } -\frac{b}{a}.$$

Hence, *if the independent term vanishes, then one of the roots vanishes*,

(*b*) If $b=0$, the equation becomes

$$ax^2+c=0$$

$$\therefore \pm\sqrt{-\frac{c}{a}}.$$

Hence, *if the co-efficient of x vanishes, the roots are equal in magnitude but opposite in sign.*

In this case the equation becomes a *pure quadratic*.

(*c*) If both $b=0$, and $c=0$, the equation becomes

$$ax^2=0$$

$$\therefore \quad x=0,\ 0.$$

Hence, *if the co-efficient of x and the independent term both vanish, then both the roots vanish.*

(*d*) If $a=0$, the equation becomes

$$0.x^2+bx+c=0$$

or, $$bx+c=0$$

that is to say, it reduces to a simple equation with a single root $x=-\dfrac{c}{b}$; what becomes of the other root?

The other root exists but its value is infinite. This can be brought out in the following manner:

Put $x=\dfrac{1}{y}$; then the equation becomes

$$a.\frac{1}{y^2}+b.\frac{1}{y}+c=0$$

or, $$cy^2+by+a=0$$

Now if $a=0$, we have $cy^2+by=0$

or, $$y(cy+b)=0$$

$$\therefore \quad y=0, \text{ or } -\frac{b}{c}.$$

Hence, since $x=\dfrac{1}{y}$, $\quad x=\infty$, or $-\dfrac{c}{b}$.

Thus, *if the co-efficient of x^2 vanishes, then one root becomes infinite,*

NOTE 1. We have already seen in Art. 11 that infinity and zero really represent not definite values but limits; hence the above result really indicates that *as the co-efficient of x^2 gets smaller and smaller one of the roots gets larger and larger, so that if the co-efficient of x^2 diminishes indefinitely one of the roots increases indefinitely.*

NOTE 2. It may be proved exactly in a similar manner that *if the co-efficient of the highest power of x in an equation of any degree vanishes then one of the roots becomes infinite.*

(*e*) If both $a=0$ and $b=0$, then the equation in y obtained above becomes $cy^2=0$. Hence, $y=0,\ 0$.

Hence, since $x=\dfrac{1}{y}$, $x=\infty,\ \infty$.

Thus, *if the co-efficients of x^2 and of x both vanish, both the roots become infinite.*

(*f*) *If both $a=0$, and $c=0$, then one root is infinite and the other zero.*

It follows at once from Case (*a*) by putting $a=0$.

(*g*) If $a=0$, $b=0$, and $c=0$, the equation becomes

$$0.x^2+0.x+0=0,$$

which is satisfied by all finite values of x. *The equation in this case is really an identity.* [See Art. 69.]

71. Sum and Product of Roots.

If the equation $ax^2+bx+c=0$ has two roots α and β, then

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a}, \text{ and } \beta=\frac{-b-\sqrt{b^2-4ac}}{2a}$$

Hence $$\alpha+\beta=\frac{-b+\sqrt{b^2-4ac}}{2a}+\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$=\frac{-2b}{2a}=-\frac{b}{a};$$

and $$\alpha\beta=\frac{(-b+\sqrt{b^2-4ac})(-b-\sqrt{b^2-4ac})}{4a^2}$$

$$=\frac{(-b)^2-(b^2-4ac)}{4a^2}=\frac{4ac}{4a^2}=\frac{c}{a}.$$

Thus we have $\alpha+\beta=-\dfrac{b}{a}$, $\alpha\beta=\dfrac{c}{a}$

That is to say, *the sum of the roots of a quadratic equation = the co-efficient of x divided by the co-efficient of x^2 with its sign changed; and the product of the roots = the independent term divided by the co-efficient of x^2.*

Cor. 1. If α, β be the roots of the quadratic equation $x^2+px+q=0$, in which the co-efficient of x^2 is unity, then

$$\alpha+\beta=-p, \text{ and } \alpha\beta=q;$$

that is, *the sum of the roots = the co-efficient of x with its sign changed; and the product of the roots = the independent term.*

Cor. 2. Reciprocal roots, and equal and opposite roots.

(i) If $c=a$ we have $\alpha\beta=\dfrac{c}{a}=\dfrac{a}{a}=1$, so that *the roots are reciprocal to each other.*

(ii) If $c=-a$, have $\alpha\beta=\frac{c}{a}=-\frac{a}{a}=-1$, so that *one root is reciprocal to the other and also opposite in sign.*

(iii) If $b=0$, we have $\alpha+\beta=\frac{b}{a}=0$, so that *its roots are equal in magnitude but opposite in sign.* [See Art. 70 (*b*).]

NOTE. *Relation between roots and co-efficients in an equation of any degree.*

Similar relations between the roots and the co-efficients can be proved for equations of any degree. For instance, let us take a *cubic equation* Let α, β, γ be the three roots. Then evidently the equation can be written in the form

$$(x-\alpha)(x-\beta)(x-\gamma)=0$$

or
$$x^3-(\alpha+\beta+\gamma)x^2+(\beta\gamma+\gamma\alpha+\alpha\beta)x-\alpha\beta\gamma=0.$$

If therefore the given cubic equation is $x^3+px^2+qx+r=0$, we have

$$\left.\begin{aligned}\alpha+\beta+\gamma&=-p\\ \beta\gamma+\gamma\alpha+\alpha\beta&=q\\ \alpha\beta\gamma&=-r\end{aligned}\right\}$$

That is to say, *if the co-efficient of x^3 is unity, the sum of the roots = the co-efficient of x^2 with the sign changed; the sum of the product of the roots taken two at a time = the co-efficient of x; and the product of the roots = the independent term with the sign changed.*

Corresponding results can be established for equations of the fourth and higher degrees.

72. Factors of a Quadratic Expression.

Since by the preceding article, if α and β are the roots of the quadratic equation $ax^2+bx+c=0$,

$$\alpha+\beta=-\frac{b}{a} \text{ and } \alpha\beta=\frac{c}{a}.$$

we have
$$\begin{aligned}ax^2+bx+c&=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)\\ &=a\{x^2-(\alpha+\beta)x+\alpha\beta\}\\ &=a(x-\alpha)(x-\beta).\end{aligned}$$

This result enables us *to factorize any quadratic expression*, such as ax^2+bx+c, for if α, β be the roots of the quadratic equation obtained by putting the quadratic expression equal to zero, $ax^2+bx+c=a(x-\alpha)(x-\beta)$.

Cor. 1. If α, β are the roots of $x^2+px+q=0$, then

$$x^2+px+q=(x-\alpha)(x-\beta).$$

Cor. 2. If the two roots of the equation $ax^2+bx+c=0$; be equal, then the quadratic expression ax^2+bx+c is a perfect square, for, in this case,

$$ax^2+bx+c=a(x-\alpha)^2.$$

73. Formation of Quadratic Equation when roots given.

Let α and β be the given roots, then it is obvious that the required equation is

$$(x-\alpha)(x-\beta)=0$$

or, $$x^2-(\alpha+\beta)x+\alpha\beta=0.$$

That is when the roots are given, *the co-efficient of x is the sum of the roots with the sign changed, and the independent term is the product of the roots, the co-efficient of x^2 being unity.*

74. Sign of the Roots of a Quadratic.

Let α and β be the roots of the quadratic equation $ax^2+bx+c=0$, so that $\alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$ [Art. 71.] Then,

(i) Both the roots will be positive, if both $\alpha+\beta$ and $\alpha\beta$ be positive, that is if both $-\dfrac{b}{a}$ and $\dfrac{c}{a}$ be positive; hence, *in order that both the roots may be positive, c and a must be of the same sign and b of the opposite sign.*

(ii) Both the roots will be negative, if $\alpha+\beta$ be negative and $\alpha\beta$ be positive, that is, if $\dfrac{b}{a}$ be negative and $\dfrac{c}{a}$ positive; hence, *in order that both the roots may be negative, a, b and c must all be of the same sign.*

(iii) One root will be positive and the other negative, if $\alpha\beta$ be negative, that is, if $\dfrac{c}{a}$ be negative; hence, *in order that one root may be positive and the other negative, c and a must be of opposite signs.*

75. Conjugate Roots.

(*i*) *In a quadratic equation with rational co-efficients, irrational roots occur in pairs, that is, if* $\alpha+\sqrt{\beta}$ *be one root, the other root will be* $\alpha-\sqrt{\beta}$.

Let $\alpha+\sqrt{\beta}$ be a root of the quadratic equation $ax^2+bx+c=0$. Then

$$a(\alpha+\sqrt{\beta})^2+b(\alpha+\sqrt{\beta})+c=0$$

or, $(a\alpha^2+a\beta+b\alpha+c)+(2a\alpha+b)\sqrt{\beta}=0.$

Hence, equating the rational and irrational parts separately, we have

$$a\alpha^2+a\beta+b\alpha+c=0 \text{ and } (2a\alpha+b)\sqrt{\beta}=0$$

$\therefore$ by subtraction, $(a\alpha^2+a\beta+b\alpha+c)-(2a\alpha+b)\sqrt{\beta}=0$

or, $$a(\alpha-\sqrt{\beta})^2+b(\alpha-\sqrt{\beta})+c=0$$

which shows that $\alpha-\sqrt{\beta}$ is also a root of the given equation.

Alternative method : Let $x=\alpha+\sqrt{\beta}$. Then $x-\alpha=\sqrt{\beta}$.

$\therefore$ $$(x-\alpha)^2=\beta.$$

or, $$x^2-2\alpha x+\alpha^2-\beta=0$$

which is a quadratic equation with rational co-efficients and one root of which is $\alpha+\sqrt{\beta}$.

Actually solving the equation we have $x=\alpha\pm\sqrt{\beta}$, which shows that if $\alpha+\sqrt{\beta}$ be one root, the other root is $\alpha-\sqrt{\beta}$.

(*ii*) *In a quadratic equation with real co-efficients, imaginary roots occur in pairs, that is, if* $\alpha+\beta\sqrt{-1}$ *be one root, the other root will be* $\alpha-\beta\sqrt{-1}$.

Let $\alpha+\beta\sqrt{-1}$ be a root of the quadratic equation $ax^2+bx+c=0$, then,

$$a(\alpha+\beta\sqrt{-1})^2+b(\alpha+\beta\sqrt{-1})+c=0$$

or, $(a\alpha^2-a\beta^2+b\alpha+c)+(2a\alpha\beta+b\beta)\sqrt{-1}=0.$

Hence, equating the real and imaginary parts separately,

$$a\alpha^2-a\beta^2+b\alpha+c=0 \text{ and } (2a\alpha\beta+b\beta)\times\sqrt{-1}=0$$

$\therefore$ by subtraction, $(a\alpha^2-a\beta^2+b\alpha+c)-(2a\alpha\beta+b\beta)\sqrt{-1}=0$

or, $$a(\alpha-\beta\sqrt{-1})^2+b(\alpha-\beta\sqrt{-1})+c=0,$$

which shows that $\alpha-\beta\sqrt{-1}$ is also a root of the given equation.

Alternative method: Let $x=\alpha+\beta\sqrt{-1}$. Then $x-\alpha=\beta\sqrt{-1}$.

Squaring, $(x-\alpha)^2=-\beta^2$.

$\therefore \quad x^2-2\alpha x+\alpha^2+\beta^2=0$

which is a quadratic equation with real co-efficients and one root of which is $\alpha+\beta\sqrt{-1}$.

Actually solving the equation, we have $x=\alpha\pm\beta\sqrt{-1}$, which shows that if $\alpha+\beta\sqrt{-1}$ be one root, the other root is $\alpha-\beta\sqrt{-1}$.

Example 1. *Examine the nature of the roots of the equations* (*i*) $10x^2-7x-12=0$, *and* (*ii*) $3x^2+7x+5=0$.

(i) Here, $a=10$, $b=-7$ and $c=-12$.

$\therefore \quad b^2-4ac=(-7)^2-4\times10(-12)=49+480=529=23^2$, which is a perfect square.

Hence the roots are real, rational and unequal.

(ii) Here $a=3$, $b=7$, and $c=5$.

$\therefore \quad b^2-4ac=7^2-4\times3\times5=49-60=-11$, a negative quantity.

Hence the roots are imaginary.

Example 2. *Find the value of* p *so that the equation* $20x^2+px+5=0$ *may have two equal roots.*

In order that the equation may have two equal roots we must have $p^2-4\times20\times5=0$, *i.e.*, $p^2=400$. $\therefore$ $p=\pm20$.

Example 3. *Prove that the roots of*

(*i*) $(a+b-c)x^2-3(a+b)x+(a+b+c)=0$ *are real.*

(*ii*) $(b+c)x^2-(a+b+c)x+a=0$ *are rational.* [*Cal.Int.*1910]

(i) $\{3(a+b)\}^2-4(a+b-c)(a+b+c)$

$$=9(a+b)^2-4\{(a+b)^2-c^2\}$$
$$=5(a+b)^2+4c^2,$$

which is obviously a positive quantity, each of $(a+b)^2$ and c^2 being positive. Hence the roots are real.

(ii) $(a+b+c)^2-4a(b+c)=a^2+(b+c)^2-2a(b+c)$

$$=(a-b-c)^2, \text{ a perfect square.}$$

Hence the roots are rational.

Example 4. *Form the equations whose roots are (i) 3 and* -5; *(ii)* $2+\sqrt{3}$ *and* $2-\sqrt{3}$; *and (iii)* $\frac{a+b}{a-b}$ *and* $\frac{a-b}{a+b}$.

(i) Since the roots are 3 and -5, the required equation is

$$(x-3)(x+5)=0,$$

or, $$x^2+2x-15=0.$$

(ii) Since $(2+\sqrt{3})+(2-\sqrt{3})=4$, and $(2+\sqrt{3})(2-\sqrt{3})$ $=4-3=1$, the required equation is $x^2-4x+1=0$.

(iii) Since $\frac{a+b}{a-b}+\frac{a-b}{a+b}=\frac{2(a^2+b^2)}{a^2-b^2}$, and $\frac{a+b}{a-b}\times\frac{a-b}{a+b}=1$,

the required equation is $$x^2-\frac{2(a^2+b^2)}{a^2-b^2}x+1=0,$$

or, $$(a^2-b^2)x^2-2(a^2+b^2)x+a^2-b^2=0.$$

Example 5. *For what values of n will the expression* $x^2-(n-1)x+n+\frac{1}{4}$ *be a perfect square?* [*Bom. P. E.* 1886]

The given expression will be a perfect square when the corresponding equation has a pair of equal roots. Hence the condition is

$$(n-1)^2=4(n+\tfrac{1}{4})$$

or, $$n^2-2n+1=4n+1$$

or, $$n^2-6n=0$$

or, $$n(n-6)=0$$

$\therefore$ $$n=0 \text{ or } 6.$$

Example 6. *Form the quadratic equation with rational co-efficients, one of whose roots is* $5+3\sqrt{2}$.

When one root is $5+3\sqrt{2}$, the other root must be $5-3\sqrt{2}$, for the co-efficients are to be rational. [Art. 75 (i).]

Now $(5+3\sqrt{2})+(5-3\sqrt{2})=10$

and $(5+3\sqrt{2})\times(5-3\sqrt{2})=25-18=7.$

Hence the required equation is $x^2-10x+7=0$.

Otherwise: Let $x=5+3\sqrt{2}$. Then $x-5=3\sqrt{2}$ $\therefore (x-5)^2=18$.

or, $x^2-10x+7=0$, the required equation.

Example 7. *Form the quadratic equation with real co-efficients, one of whose roots is* $3+\sqrt{-5}$.

Since, from Art. 75 (ii), the other root must be $3-\sqrt{-5}$,

we have $$(3+\sqrt{-5})+(3-\sqrt{-5})=6,$$

and $$(3+\sqrt{-5})(3-\sqrt{-5})=9+5=14.$$

$\therefore$ the required equation is $x^2-6x+14=0$.

Otherwise: Let $x=3+\sqrt{-5}$. Then $x-3=\sqrt{-5}$,

$\therefore$ $(x-3)^2=-5$, or, $x^2-6x+14=0$, the required equation.

Example 8. *If r be the ratio of the roots of the equation $ax^2+bx+c=0$, show that* $\dfrac{(r+1)^2}{r}=\dfrac{b^2}{ac}$. [*Cal. Int.* 1934]

Let α be one of the roots, so that $r\alpha$ is the other root.

$$\left.\begin{aligned} &\text{Then} \quad \alpha+r\alpha=-\frac{b}{a}, \quad i.e., \; \alpha(1+r)=-\frac{b}{a} \cdots (1) \\ &\text{and} \quad \alpha.r\alpha=\frac{c}{a}, \quad i.e., \quad \alpha^2=\frac{c}{ar} \cdots (2) \end{aligned}\right\} \quad \text{[Art. 71.]}$$

From (1), $\alpha=-\dfrac{b}{a(1+r)}$, and substituting this value of α in (2)

we have $\dfrac{b^2}{a^2(1+r)^2}=\dfrac{c}{ar}$, whence $\dfrac{(r+1)^2}{r}=\dfrac{b^2}{ac}$, which is the condition required.

Example 9. *Find the condition that* $\dfrac{a}{x-a}+\dfrac{b}{x-b}=k$ *may have its roots equal in magnitude but opposite in sign.*

The eqn. is $$\frac{a}{x-a}+\frac{b}{x-b}=\frac{(a+b)x-2ab}{x^2-(a+b)x+ab}=k$$

or, $$kx^2-k(a+b)x+kab=(a+b)x-2ab$$

or, $$kx^2-(k+1)(a+b)x+(k+2)ab=0.$$

$\therefore$ the required condition is $(k+1)(a+b)=0$ [Art. 70 (*b*)]

either $k=-1$, or $a+b=0$.

Example 10. *Show that the values of x obtained from the equations $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and $y=mx+c$ will be equal, if $c^2=m^2a^2+b^2$.*

Since $y=mx+c$, substituting this value of y in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

we have $$\frac{x^2}{a^2}+\frac{(mx+c)^2}{b^2}=1$$

or, $a^2(mx+c)^2+b^2x^2=a^2b^2$

or, $(m^2a^2+b^2)x^2+2mca^2x+a^2(c^2-b^2)=0.$

The roots of this equation will be equal,

if $4m^2c^2a^4-4a^2(c^2-b^2)(m^2a^2+b^2)=0$

or, $m^2c^2a^2-(c^2-b^2)(m^2a^2+b^2)=0$ [dividing by $4a^2$]

or, $-b^2c^2+m^2a^2b^2+b^4=0$

or, $c^2=m^2a^2+b^2$ [dividing by b^2]

NOTE. The geometrical interpretation is that the straight line $y=mx+c$ touches the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, if $c^2=m^2a^2+b^2$.

EXERCISE 24

1. Examine the nature of the roots of the following equations :

(i) $x^2+4x+1=0.$ (ii) $2x^2+7x+12=0.$
(iii) $10x^2-11x+3=0.$ (iv) $21x^2+5x-6=0.$
(v) $4x^2-28x+49=0.$ (vi) $9x^2+30x+25=0.$
(vii) $ax^2+(a+b)x+b=0.$ (viii) $ax^2+(1+ab)x+ab=0.$
(ix) $x^2-ax+b^2=0$ $(a>2b)$. (x) $ax^2-bx=c(ac-b).$

2. Form the quadratic equations whose roots are :

(i) 2, 3. (ii) 7, -4. (iii) $\frac{2}{3}, -\frac{3}{5}$. (iv) $7\pm4\sqrt{3}$.
(v) $\sqrt{3}\pm1$. (vi) $\frac{1}{3}(5\pm3\sqrt{2})$. (vii) $a\pm\sqrt{b}$. (viii) $a\pm b$.

(ix) $\frac{m-n}{m+n}, -\frac{m+n}{m-n}$. (x) $-\frac{a}{a+b}, \frac{b}{a+b}$.

3. Form the quadratic equations having rational co-efficients, one of whose roots is

(i) $3+\sqrt{7}$. (ii) $\sqrt{3}-1$. (iii) $5\sqrt{-1}-3$.

(iv) $5+\sqrt{-2}$. (v) $a+b+\sqrt{(a^2+b^2)}$. (vi) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$.

(vii) $\frac{1}{2}(-1+\sqrt{-3})$. (viii) $\sqrt{5}+1$. [Cal. F.A. 1876]

4. For what value of k will the roots of the following equations be equal :

(i) $3x^2+kx+12=0$; (ii) $16x^2-kx+3=0$;
(iii) $x^2+2(1+k)x+k^2=0$?

5. If $a+b+c=0$, prove that the roots of $ax^2+bx+c=0$ are rational.

6. Prove that the roots of the equation $(a-b+c)x^2+4(a-b)x+(a-b-c)=0$ are real.

7. Show that the equation $5ax^2-(3a+5b)x+3b=0$ has rational roots.

8. Show that the roots of $(x-a)(x-b)=c^2$ are always rea

9. If the roots of $ax^2+bx+c=0$ are real, prove that the roots of $ax^2+(b-1)x+c=\dfrac{2b-1}{4a}$ are also real.

10. If the roots of $ax^2+2bx+c=0$ are imaginary, prove that the roots of $ax^2+2(a+b)x+a+2b+c=0$ are also imaginary.

11. For what values of m will the equation

$$x^2-2(5+2m)x+3(7+10m)=0$$

have (i) equal roots and (ii) reciprocal roots? [Cal. Int. 1936]

12. The roots of $(c^2-ab)x^2-2(a^2-bc)x+(b^2-ac)=0$ are equal; show that either $a=0$ or $a^3+b^3+c^3=3abc$.

13. If α be a root of the equation $4x^2+2x-1=0$, prove that $4\alpha^3-3\alpha$ is the other root. [Bom. P.E. 1889]

14. Prove, without solving, that the positive root of $x^2-8x-8=0$ is greater than 8. [Bom. P.E. 1893]

15. Prove that the roots of $(b+c-2a)x^2+(c+a-2b)x+(a+b-2c)=0$ are rational.

16. If $a+b+c=0$, prove that the roots of $(b+c-a)x^2+(c+a-b)x+(a+b-c)=0$ are rational.

17. Show that if the roots of the equation $(b^2+b_1{}^2)x^2+2(ab+a_1b_1)x+a^2+a_1{}^2=0$ be real, they will be equal.

18. If one root of the equation $ax^2+bx+c=0$ be four times the other, show that $4b^2=25ac$. [Cal. Int. 1940]

19. If the roots of the equation $ax^2+bx+c=0$ bear to one another the ratio 3 : 4, prove that $12b^2=49ac$. [Cal. Int. 1945]

20. If one root of the equation $x^2-px+q=0$ be twice the other, show that $2p^2=9q$. [Cal. Int. 1937]

21. If one root of the equation $ax^2+bx+c=0$ be the square of the other, prove that $b^3+a^2c+ac^2=3abc$.

[Cal. F.A. 1879]

22. If one root of the equation $x^2+px+q=0$ is the square of the other, prove that $p^3-q(3p-1)+q^2=0$. [Cal. Int. 1943]

23. Form a quadratic equation with rational co-efficients, one of whose roots is $\frac{\sqrt{a}}{\sqrt{a}+\sqrt{(a-b)}}$. What is the other root?

24. Obtain a quadratic equation with rational co-efficients, of which one root is $\frac{p-\sqrt{p^2-4q}}{p+\sqrt{p^2-4q}}$. [Cal. Int. 1927]

25. Prove that, if one of the equations $ax^2-(3a-b)x+b=0$ and $ax^2-(5a-b)x+4a=0$ has equal roots, so has the other.

26. Prove that the roots of one of the two equations $ax^2-2bx+a=0$ and $bx^2-2ax+b=0$ must be imaginary.

27. Show that the roots of $bx^2+(b-c)x=c+a-b$ are real, if those of $ax^2+b(2x+1)=0$ are imaginary.

28. If the roots of $\frac{1}{x}+\frac{1}{x+a}=\frac{1}{m}+\frac{1}{m+a}$ are equal in magnitude but opposite in sign, show that $a^2=2m^2$.

29. If the ratio of the roots of the equation $x^2+px+q=0$ be equal to that of the roots of $x^2+p_1x+q_1=0$, show that $p^2q_1=p_1{}^2q$. [Cal. F.A. 1885]

30. Prove that the roots of $ax^2+bx+c=0$ will be rational, if $b=ka+\frac{c}{k}$, where k is any rational quantity.

31. For what value of k will the equation

$$\frac{a}{x+a+k}+\frac{b}{x+b+k}=1$$

have two equal roots with opposite signs?

32. If the roots of $lx^2+nx+n=0$ be in the ratio $p:q$, prove that $$\sqrt{\frac{p}{q}}+\sqrt{\frac{q}{p}}+\sqrt{\frac{n}{l}}=0.$$

33. Show that the two values of x obtained from the equations $y^2=4ax$ and $y=mx+c$ will be equal if $cm=a$.

34. Show that the two values of x obtained from the equations $ax^2+by^2=1$ and $ax+by=1$ will be equal if $a+b=1$.

35. Show that the arithmetic mean of the roots of $x^2-2ax+b^2=0$ is the geometric mean of the roots of $x^2-2bx+a^2=0$, and *vice versa*. [Cal. F.A. 1887]

76. Symmetrical Functions of the Roots.

With the help of the results obtained in Art. 71, any symmetrical function of the roots α and β of a quadratic equation can be expressed in terms of the co-efficients of the quadratic.

Rule. *Express the symmetrical function of the roots in terms of $\alpha+\beta$ and $\alpha\beta$, and then use the results*

$$\alpha+\beta=-\frac{b}{a}, \text{ and } \alpha\beta=\frac{c}{a}.$$

77. We have shown in Art. 73 how a quadratic equation can be formed whose roots are given. In a similar manner we can form the quadratic equation whose roots bear some symmetrical relation to those of a given quadratic equation.

The method is illustrated in the following examples.

Example 1. *If α and β be the roots of $ax^2+bx+c=0$, find the values of (i) $\alpha^2+\beta^2$, (ii) $\alpha^3+\beta^3$, (iii) $(m\alpha+n\beta)(n\alpha+m\beta)$, and (iv) $\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}$.*

Since $\alpha+\beta=-\frac{b}{a}$, and $\alpha\beta=\frac{c}{a}$, we have

(i) $$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\left(-\frac{b}{a}\right)^2-2.\frac{c}{a}$$

$$=\frac{b^2}{a^2}-\frac{2c}{a}=\frac{b^2-2ac}{a^2}$$

(ii) $$\alpha^3+\beta^3=(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)=\left(-\frac{b}{a}\right)^3-3.\frac{c}{a}\left(-\frac{b}{a}\right)$$

$$=\frac{3bc}{a^2}-\frac{b^3}{a^3}=\frac{b(3ac-b^2)}{a^3}$$

(iii) $$(m\alpha+n\beta)(n\alpha+m\beta)=mn\alpha^2+(m^2+n^2)\alpha\beta+mn\beta^2$$
$$=mn(\alpha+\beta)^2+(m-n)^2\alpha\beta$$

$$=mn\frac{b^2}{a^2}+(m-n)^2\frac{c}{a}$$

$$=\frac{mnb^2+(m-n)^2ac}{}$$

(iv) $$\frac{\alpha^2}{\beta}+\frac{\beta^2}{\alpha}=\frac{\alpha^3+\beta^3}{\alpha\beta}=\frac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha\beta}$$

$$=\frac{\left(-\frac{b}{a}\right)^3-3\frac{c}{a}\left(-\frac{b}{a}\right)}{\frac{c}{a}}$$

$$=\frac{b(3ac-b^2)}{a^2c}.$$

Example 2. *If α and β be the roots of the equation $x^2+px+q=0$, find the value of (i) $\alpha-\beta$, (ii) $\alpha^4+\beta^4$, and (iii) $\alpha^3\beta+\alpha\beta^3$.*

Since $\alpha+\beta=-p$ and $\alpha\beta=q$, we have

(i) $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=p^2-4q$. $\therefore\ \alpha-\beta=\pm\sqrt{p^2-4q}$.

(ii) $\alpha^4+\beta^4=(\alpha^2+\beta^2)^2-2\alpha^2\beta^2=\{(\alpha+\beta)^2-2\alpha\beta\}^2-2\alpha^2\beta^2$
$=(p^2-2q)^2-2q^2=p^4-4p^2q+2q^2$.

(iii) $\alpha^3\beta+\alpha\beta^3=\alpha\beta\{(\alpha+\beta)^2-2\alpha\beta\}=q(p^2-2q)$.

Example 3. *If α and β be the roots of $ax^2+bx+c=0$, find the value of* $\dfrac{1}{a\alpha+b}+\dfrac{1}{a\beta+b}$.

Since α is a root of $ax^2+bx+c=0$, we have $a\alpha^2+b\alpha+c=0$.

$\therefore$ $$\alpha(a\alpha+b)=-c, \text{ i.e., } \frac{1}{a\alpha+b}=-\frac{\alpha}{c}.$$

Similarly, $$\frac{1}{a\beta+b}=-\frac{\beta}{c}.$$

by addition, we have

$$\frac{1}{a\alpha+b}+\frac{1}{a\beta+b}=-\frac{\alpha}{c}-\frac{\beta}{c}=-\frac{\alpha+\beta}{c}=\frac{b}{ac}\cdot\left[\because\ \alpha+\beta=-\frac{b}{a}.\right]$$

Example 4. *If α and β be the roots of $ax^2+bx+c=0$, find the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.* [*Cal. Int.* 1936]

Since $\alpha+\beta=-\dfrac{b}{a}$ and $\alpha\beta=\dfrac{c}{a}$, we have the sum of the roots of the required equation

$$=\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{\alpha^2+\beta^2}{\alpha\beta}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha\beta}=\frac{b^2-2ac}{ac};$$

and the product of the roots $=\frac{\alpha}{\beta}\times\frac{\beta}{\alpha}=1.$

Hence the required equation is

$$x^2-\frac{b^2-2ac}{ac}x+1=0$$

or, $$acx^2-(b^2-2ac)x+ac=0.$$

Example 5. *If α and β be the roots of $ax^2+bx+c=0$, form the equation whose roots are $(\alpha+\beta)^2$ and $(\alpha-\beta)^2$.* [*Cal.Int.*1924]

Since $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$, the sum of the roots of the required equation $=(\alpha+\beta)^2+(\alpha-\beta)^2=2(\alpha^2+\beta^2)$

$$=2\{(\alpha+\beta)^2-2\alpha\beta\}$$

$$=2\left(\frac{b^2}{a^2}-\frac{2c}{a}\right)=\frac{2(b^2-2ac)}{a^2}.$$

and the product of the roots $=(\alpha+\beta)^2(\alpha-\beta)^2$

$$=(\alpha+\beta)^2\{(\alpha+\beta)^2-4\alpha\beta\}$$

$$=\frac{b^2}{a^2}\left(\frac{b^2}{a^2}-\frac{4c}{a}\right)=\frac{b^2(b^2-4ac)}{a^4}.$$

Hence the required equation is

$$x^2-\frac{2(b^2-2ac)}{a^2}x+\frac{b^2(b^2-4ac)}{a^4}=0.$$

or, $$a^4x^2-2a^2(b^2-2ac)x+b^2(b^2-4ac)=0.$$

Example 6. *If α and β be the roots of $x^2-px+q=0$, form the equation whose roots are $m\alpha+n\beta$ and $n\alpha+m\beta$.*

We have $\alpha+\beta=p$, and $\alpha\beta=q$.

$\therefore$ the sum of the roots of the required equation

$$=(m\alpha+n\beta)+(n\alpha+m\beta)=(m+n)(\alpha+\beta)=(m+n)p\ ;$$

and the product of the roots $=(m\alpha+n\beta)(n\alpha+m\beta)$

$$=mn(\alpha^2+\beta^2)+(m^2+n^2)\alpha\beta$$

$$=mn(\alpha+\beta)^2+(m-n)^2\alpha\beta$$

$$=mnp^2+(m-n)^2q.$$

Hence the required equation is

$$x^2-(m+n)px+mnp^2+(m-n)^2q=0.$$

Example 7. *If α, β be the roots of the equation $x^2+x+1=0$, form the equation whose roots are α^2 and β^2. Explain why you get the same equation as the given one.* [*All. I.E.* 1924]

We have $\alpha+\beta=-1$, and $\alpha\beta=1$.

$\therefore\ \alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=1-2=-1$ and $\alpha^2\beta^2=(\alpha\beta)^2=1$.

Hence the required equation is $x^2+x+1=0$, which is the same as the given equation.

Now since $\alpha+\beta=-1$ we have $\beta=-\alpha-1$. ... (1)

Also since α is a root of $x^2+x+1=0$, we have $\alpha^2+\alpha+1=0$

$\therefore$ $\alpha^2=-\alpha-1$ ... (2)

$\therefore$ from (1) and (2), $\alpha^2=\beta$; similarly it can be shown that $\beta^2=\alpha$.

Thus, α and β are equal to β^2 and α^2 respectively, and therefore we get the same equation.

Example 8. *If $x=\frac{1}{2}(-1+\sqrt{-23})$, find the value of* $5x^3+8x^2+33x+25$.

Let us form the quadratic equation one of whose roots is $\frac{1}{2}(-1+\sqrt{-23})$. Hence the other root is its conjugate $\frac{1}{2}(-1-\sqrt{-23})$. [Art. 75 (ii).]

Hence, the sum of the roots $=-1$, and the product of the roots $=\frac{1}{4}(1+23)=6$; so that the required quadratic equation is

$$x^2+x+6=0 \qquad \text{... (1)}$$

Now, the given expression $=5x(x^2+x+6)+3(x^2+x+6)+7$

$$=5x\times 0+3\times 0+7 \qquad \text{[from (1)]}$$

$$=7.$$

EXERCISE 25.

1. If α and β be the roots of the equation $ax^2+bx+c=0$, find the values of :

(i) $\alpha-\beta$. (ii) $\alpha^2-\beta^2$. (iii) $\alpha^3-\beta^3$.

(iv) $\alpha^2\beta+\alpha\beta^2$. (v) $\alpha^4+\beta^4$. (vi) $\alpha^4+\alpha^2\beta^2+\beta^4$.

(vii) $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$. (viii) $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$. (ix) $\frac{\alpha^2}{\beta}-\frac{\beta^2}{\alpha}$.

2. If α and β be the roots of the equation $x^2-px+q=0$, find the values of :

(i) $(\alpha+2\beta)(2\alpha+\beta)$. (ii) $\dfrac{1}{\alpha^3}+\dfrac{1}{\beta^3}$. [Cal. F.A. 1886]

(iii) $\dfrac{\alpha}{\alpha^2+q}+\dfrac{\beta}{\beta^2+q}$. (iv) $\dfrac{\alpha^2}{\beta^2}+\dfrac{\beta^2}{\alpha^2}$. [Bom. P.E. 1885]

3. If α and β be the roots of the equation $ax^2+2bx+c=0$, determine $\alpha^3+\beta^3$ and $\alpha^5+\beta^5$ in terms of a, b and c. [Cal. F.A. 1893]

4. If α and β be the roots of $ax^2+bx+c=0$, find the value of $\dfrac{1}{(a\alpha+b)^3}+\dfrac{1}{(a\beta+b)^3}$. [Cal. Int. 1943]

5. If α and β be the roots of $x^2-(1+k^2)x+\frac{1}{2}(1+k^2+k^4)=0$, show that $\alpha^2+\beta^2=k^2$. [Cal. Int. 1909]

6. If α, β and α', β' be the roots of $x^2-px+q=0$ and $x^2-p'x+q'=0$ respectively, find the value of

$(\alpha-\alpha')^2+(\beta-\alpha')^2+(\alpha-\beta')^2+(\beta-\beta')^2$. [Cal. Int. 1913]

7. If α and β be the roots of $ax^2+bx+c=0$, form the equations whose roots are

(i) α^2, β^2. (ii) α^3, β^3. (iii) $\alpha^2+\beta^2, (\alpha+\beta)^2$.

(iv) $\dfrac{1}{\alpha^2}, \dfrac{1}{\beta^2}$. (v) $m\alpha+\beta, m\beta+\alpha$. (vi) $\dfrac{1-\alpha}{1+\alpha}, \dfrac{1-\beta}{1+\beta}$.

8. If α and β be the roots of $x^2-px+q=0$, construct the equations whose roots are

(i) $\alpha+\beta, \alpha\beta$. (ii) $2\alpha-\beta, 2\beta-\alpha$.

(iii) $\alpha^2+\beta, \beta^2+\alpha$. (iv) $\alpha+\dfrac{1}{\beta}, \beta+\dfrac{1}{\alpha}$.

(v) $\dfrac{\alpha}{\alpha^2+q}, \dfrac{\beta}{\beta^2+q}$. (vi) $\dfrac{p\alpha^2}{\beta^2+q\beta}, \dfrac{p\beta^2}{\alpha^2+q\alpha}$.

9. If α and β be the roots of $x^2+px+q=0$, form the equation whose roots are $\alpha^2+\alpha\beta$ and $\beta^2+\alpha\beta$. [Bom. P.E. 1887]

10. Form an equation whose roots are the reciprocals of the roots of $x^2-3x+1=0$. [All. I.E. 1925]

11. If p and q be the roots of the equation $2x^2-5x+2=0$, find the equation whose roots are $p+q$ and $\frac{1}{2}pq$. [Cal. Int. 1919]

12. If p and q are the roots of the equation $x^2+7x+12=0$, find the equation whose roots are $(p+q)^2$ and $(p-q)^2$. [Cal. Int. 1920 and 1945]

13. If the roots of $x^2+Px+q=0$ be α, β, and the roots of $x^2+px+Q=0$ be γ, δ, find the roots of $x^2+px+q=0$ in terms of α, β, γ, δ. [Cal. Int. 1914]

14. If p and q are the roots of the equation $3x^2+6x+2=0$, find the equation whose roots are $-\dfrac{p^2}{q}$ and $-\dfrac{q^2}{p}$.

[Cal. Int. 1942]

15. Find the relation between the co-efficients a, b, c so that the sum of the squares of the roots of the equation $ax^2+bx+c=0$ may be equal to their product.

16. If α and β be the roots of the equation $x^2+px+q=0$, form the equation whose roots are $\dfrac{q}{\alpha}$ and $\dfrac{q}{\beta}$. Why is the new equation the same as the old one?

17. If α, β be the roots of the equation $x^2-px+q=0$, form the equation whose roots are $\dfrac{q}{p-\alpha}$ and $\dfrac{q}{p-\beta}$. Explain why you get the same equation as the given one.

18. Show that the roots of $x^2-(p^2-3q^2)x+q^2(p^2-4q^2)=0$ are the product, and the square of the difference, of the roots of the equation $x^2-px+q^2=0$. [Cal. F.A. 1877]

19. Form the equation whose roots are the squares of the sum and of the difference of the roots of $2x^2+2(m+n)x+m^2+n^2=0$. [All. I.E. 1928]

20. If α and β be the roots of the equation $x^2+2ax+b=0$, form a quadratic equation with rational co-efficients, one of whose roots is $\alpha+\beta+\sqrt{(\alpha^2+\beta^2)}$. [Cal. F.A. 1878]

21. If $\alpha\pm\sqrt{\beta}$ be the roots of the equation $x^2+px+q=0$, prove that $\dfrac{1}{\alpha}\pm\dfrac{1}{\sqrt{\beta}}$ will be the roots of the equation

$$(p^2+4q)(p^2x^2+4px)=16q.$$ (Cal. F.A. 1880]

22. If α, β be the roots of $ax^2+bx+c=0$, show that the equation whose roots are $\dfrac{1}{\alpha+\beta}$ and $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$ is $bcx^2+(b^2+ca)x+ab=0$. [Bom. P.E. 1882 and All. I.E. 1892]

23. If α, β be the roots of $x^2-px+q=0$, find the condition that the roots may be connected by the relation $m\alpha+n\beta=1$.

24. If the roots of the equation $ax^2+2bx+c=0$ be α and β, and those of the equation $Ax^2+2Bx+C=0$ be $\alpha+\delta$ and $\beta+\delta$, prove that $\frac{b^2-ac}{B^2-AC}=\left(\frac{a}{A}\right)^2$. [Cat. Int. 1912]

25. If the difference of the roots of the equation $x^2-px+q=0$ be the same as that of the equation $x^2-qx+p=0$, show that $p+q+4=0$, unless $p=q$. [Cal. Int. 1941]

26. If the roots of $x^2+2px+q=0$ and $x^2+2qx+p=0$ differ by a constant, show that $p+q+1=0$. [Cal. Int. 1944]

27. If the roots of $ax^2+bx+c=0$ are the reciprocals of those of $a'x^2+b'x+c'=0$, show that $ab'=bc'$, and $aa'=cc'$.

28. If α, β are the roots of $x^2+px+q=0$, and γ, δ those of $x^2+p'x+q'=0$, find the equation whose roots are $\alpha\gamma+\beta\delta$ and $\alpha\delta+\beta\gamma$.

29. If the difference of the roots of the equation $x^2-px+q=0$ be unity, prove that $p^2+4q^2=(1+2q)^2$. [All. I.E. 1917].

30. In copying a quadratic equation of the form $x^2+px+q=0$, the co-efficient of x was incorrectly written as -10 instead of -11, and the roots were thus found to be 4 and 6. Find the correct roots of the equation in the book.

31. In copying a quadratic equation of the form $x^2+px+q=0$, a mistake was made in writing out the co-efficient of x and the roots were found to be 3 and 10; in a fresh attempt the independent term was incorrectly written and the roots were found to be 4 and 7. Find the roots of the correct equation.

32. Find the value of the expression $x^3-7x^2+13x-2$, for $x=2+\sqrt{3}$. [All. I.E. 1929]

33. Find the value of $a^4+3a^3b+4a^2b^2+3ab^3+b^4$ when a and b are the other two roots of the equation $4x^3+(x-1)^2+2(x^2-1)=4$, besides the root $x=1$. [Cal. F.A. 1871]

78. Roots found by Inspection.

If one of the roots of a quadratic equation can be found by inspection, the other root can be found, without solving the equation, by the help of either of the results

$$\alpha+\beta=-\frac{b}{a} \text{ and } \alpha\beta=\frac{c}{a}.$$

For, if one root α is known, from the first relation the other root $\beta=-\frac{b}{a}-\alpha$, and from the second relation $\beta=\frac{c}{a}\div\alpha$.

Example 1. *Solve* $(x-a)(x-b)=(m-a)(m-b)$.

By inspection one of the roots $=m$.

Simplifying the equation we have

$$x^2-(a+b)x-m^2+(a+b)m=0.$$

Here the sum of the roots $=a+b$, and one root is m.

$\therefore$ the other root $=a+b-m$.

Example 2. *Solve* $(b+c-2a)x^2+(c+a-2b)x+(a+b-2c)=0$.

Since the sum of the co-efficients

$$(b+c-2a)+(c+a-2b)+(a+b-2c)=0,$$

it is evident that one root is $x=1$. [See Art. 5]

The product of the roots $=\dfrac{a+b-2c}{b+c-2a}$.

Hence, the other root $=\dfrac{a+b-2c}{b+c-2a}\div 1=\dfrac{a+b-2c}{b+c-2a}$.

EXERCISE 26.

Solve the following equations (by inspection):

1. $(x-3)(x-5)=(a-3)(a-5)$.

2. $(b-c)x^2+(c-a)x+(a-b)=0$.

3. $(b-c)x^2+a(c-a)x+a^2(a-b)=0$.

4. $a(x^2+1)=x(a^2+1)$. 5. $x-x^{-1}=a-a^{-1}$.

6. $(a+1)(x^2+x+1)=(x+1)(a^2+a+1)$.

7. $(x-a+b)(x+c-a)=bc$. 8. $(a+c)(x^2+b)=(x+c)(a^2+b)$.

9. $\dfrac{a}{x-a}+\dfrac{b}{x-b}=\dfrac{a}{b}+\dfrac{b}{a}$. 10. $\dfrac{b+c}{x-a}+\dfrac{c+a}{x-b}=2$.

11. $\dfrac{a+c}{x+a}+\dfrac{b+c}{x+b}=\dfrac{2(a+b+c)}{x+a+b}$.

12. $\sqrt{(x^2+ax-1)}+\sqrt{(x^2+bx-1)}=\sqrt{a}+\sqrt{b}$ [Cal.F.A.1898.]

13. If $a+b+c=0$, show that the roots of the equation $(b+c-a)x^2+(c+a-b)x+(a+b-c)=0$ are 1 and $\dfrac{a+b-c}{b+c-a}$.

14. If $a+b+c=0$, find the two roots of the equation $ax^2+bx+c=0$ in their simplest form.

15. Find for what other value of x, the expression $\dfrac{1}{x+a}+\dfrac{1}{x+b}$ will have the same value as it has when $x=c$.

79. Maximum and Minimum values.

Sometimes it is required to find out for what real values of the variable, a certain function of x has its greatest or its least value. When the function of x, after gradually increasing, ceases to increase and then begins to diminish as x increases, it is said to attain its *maximum* value ; and when the function, after gradually diminishing, ceases to diminish and then begins to increase, it is said to attain *minimum* value.

We shall now show in the next article how the maximum and minimum values of a quadratic expression, when they exist, for real values of x, can be found.

80. Maximum and Minimum values of a Quadratic Expression.

Let ax^2+bx+c be the quadratic expression.

We have $ax^2+bx+c=a\left(x^2+\frac{b}{a}x+\frac{c}{a}\right)$

$$=a\left\{\left(x+\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}\right\}$$

$$=a\left\{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right\}$$

$$=a\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a}.$$

First, let a be positive.

Then, since $\left(x+\frac{b}{2a}\right)^2$ is always positive for all real values of x, and a is positive, $a\left(x+\frac{b}{2a}\right)^2$ must be positive.

Therefore ax^2+bx+c is greater than $\frac{4ac-b^2}{4a}$ for all real values of x, except when $x+\frac{b}{2a}=0$, *i. e.*, $x=-\frac{b}{2a}$, in which case ax^2+bx+c is equal to $\frac{4ac-b^2}{4a}$.

Hence $\frac{4ac-b^2}{4a}$ is the *minimum* value of the expression ax^2+bx+c, and this value is attained when $x=-\frac{b}{2a}$.

Since $a\left(x+\frac{b}{2a}\right)^2$ increases without limit as x increases, there is no upper limit to the value of the expression and therefore it has no *maximum value.*

Secondly, let a be negative.

Here since $\left(x+\frac{b}{2a}\right)^2$ is always positive for all real values of x, and a is negative, $a\left(x+\frac{b}{2a}\right)^2$ must be negative.

Therefore ax^2+bx+c is less than $\frac{4ac-b^2}{4a}$ for all real values of x, except when $x+\frac{b}{2a}=0$, *i.e.*, $x=-\frac{b}{2a}$, in which case ax^2+bx+c is equal to $\frac{4ac-b^2}{4a}$.

Hence $\frac{4ac-b^2}{4a}$ is the *maximum* value of the expression ax^2+bx+c, and this value is attained when $x=-\frac{b}{2a}$.

Since $a\left(x+\frac{b}{2a}\right)^2$ decreases without limit as x increases, there is no lower limit to the value of the expression, and therefore it has *no minimum value.*

Otherwise :

Let $y=ax^2+bx+c$, so that $ax^2+bx+(c-y)=0$.

Solving the equation as a quadratic in x, we have

$$x=\frac{-b\pm\sqrt{b^2-4a(c-y)}}{2a}=\frac{-b\pm\sqrt{4ay-(4ac-b^2)}}{2a}$$

$$=\frac{-b\pm\sqrt{4a\left(y-\frac{4ac-b^2}{4a}\right)}}{2a}.$$

Since x is to be real, the expression $4a\left(y-\frac{4ac-b^2}{4a}\right)$ under the radical sign must be either positive or equal to zero. If it is equal to zero, $y=\frac{4ac-b^2}{4a}$; if it is positive, the factors a and $y-\frac{4ac-b^2}{4a}$ must be either both positive or both negative. Hence,

(i) if a is positive, $y-\frac{4ac-b^2}{4a}$ must also be positive, that is, y must be greater than $\frac{4ac-b^2}{4a}$. Thus, for real values of x, y is either equal to or greater than $\frac{4ac-b^2}{4a}$.

Hence the minimum value of y is $\frac{4ac-b^2}{4a}$, and the corresponding value of $x=-\frac{b}{2a}$.

(ii) If a is negative, $y-\frac{4ac-b^2}{4a}$ must also be negative, that is, y must be less than $\frac{4ac-b^2}{4a}$. Thus for real values of x, y is either equal to or less than $\frac{4ac-b^2}{4a}$.

Hence the maximum value of y is $\frac{4ac-b^2}{4a}$ and the corresponding value of $x=-\frac{b}{2a}$.

Thus we see that for real values of x, the values of the quadratic expression ax^2+bx+c vary from $-\infty$ to $\frac{4ac-b^2}{4a}$, the maximum value being $\frac{4ac-b^2}{4a}$, when a is negative; and from $\frac{4ac-b^2}{4a}$ to $+\infty$, the minimum value being $\frac{4ac-b^2}{4a}$, when a is positive; and when the expression is either maximum or minimum, the corresponding value of x is $-\frac{b}{2a}$.

Example 1. *Find the maximum value of* $-2x^2+7x+15$, *when* x *is real.*

$$-2x^2+7x+15=-2(x^2-\tfrac{7}{2}x-\tfrac{15}{2})=-2\{(x-\tfrac{7}{4})^2-\tfrac{49}{16}-\tfrac{15}{2}\}$$
$$=-2(x-\tfrac{7}{4})^2+\tfrac{169}{8}.$$

Now, since $(x-\frac{7}{4})^2$ is always positive and therefore $-2(x-\frac{7}{4})^2$ is negative, the given expression cannot be greater than $\frac{169}{8}$, and it is equal to $\frac{169}{8}$, when $x-\frac{7}{4}=0$ *i.e.*, $x=\frac{7}{4}$.

Hence the maximum value of $-2x^2+7x+15$ is $\frac{169}{8}$ and the corresponding value of $x=\frac{7}{4}$.

Otherwise: Let $y=-2x^2+7x+15$ $\therefore$ $2x^2-7x+(y-15)=0$.

$$\therefore \qquad x=\frac{7\pm\sqrt{49-8(y-15)}}{4}=\frac{7\pm\sqrt{8(\frac{169}{8}-y)}}{4}.$$

Now in order that x may be real, $\frac{169}{8}-y$ under the radical sign must be either zero or positive.

$\therefore$ y is either equal to or less than $\frac{169}{8}$.

Hence the maximum value of y is $\frac{169}{8}$, and then $x=\frac{7}{4}$.

Example 2. *Find the minimum value of* $3x^2-5x+7$ *for real values of* x.

$$3x^2-5x+7=3(x^2-\tfrac{5}{3}x+\tfrac{7}{3})$$
$$=3\{(x-\tfrac{5}{6})^2+\tfrac{7}{3}-\tfrac{25}{36}\}$$
$$=3(x-\tfrac{5}{6})^2+\tfrac{59}{12}.$$

Now, since $3(x-\frac{5}{6})^2$ is always positive for real values of x, the given expression cannot be less than $\frac{59}{12}$, and it is equal to $\frac{59}{12}$ when $x-\frac{5}{6}=0$ *i. e.*, when $x=\frac{5}{6}$. Hence the minimum value of $3x^2-5x+7$ is $\frac{59}{12}$, and the corresponding value of x is $\frac{5}{6}$.

Example 3. *If* x *be real, prove that the value of* $\dfrac{(x-1)(x+3)}{(x-2)(x+4)}$ *does not lie between* $\frac{4}{9}$ *and* 1. [*Mad. F. A.* 1884]

Let $$y=\frac{(x-1)(x+3)}{(x-2)(x+4)}=\frac{x^2+2x-3}{x^2+2x-8}.$$

Then $$(y-1)x^2+2(y-1)x+(3-8y)=0.$$

$$\therefore \qquad x=\frac{-(y-1)\pm\sqrt{(y-1)^2-(y-1)(3-8y)}}{y-1}.$$

The expression under the radical sign
$$=(y-1)^2-(y-1)(3-8y)=(y-1)(9y-4)=9(y-\tfrac{4}{9})(y-1).$$

In order that x may be real, $9(y-\frac{4}{9})(y-1)$ must be either zero or positive. When $9(y-\frac{4}{9})(y-1)=0$, we have $y=\frac{4}{9}$ or 1; when $9(y-\frac{4}{9})(y-1)$ is to be positive, the factors $y-\frac{4}{9}$ and $y-1$ must be either both positive or both negative.

Now, for any value of y less than $\frac{4}{9}$, both $y-\frac{4}{9}$ and $y-1$ are negative, and for any value of y greater than 1, both $y-\frac{4}{9}$ and $y-1$ are positive, but for any value of y lying between $\frac{4}{9}$ and 1, $y-\frac{4}{9}$ is positive and $y-1$ is negative. Hence y cannot lie between $\frac{4}{9}$ and 1.

Hence, for real values of x, the values of the given expression range from $-\infty$ to $\frac{4}{9}$ and from 1 to $+\infty$. Thus, $\frac{4}{9}$ is the maximum value and 1 the minimum value of the expression, the corresponding values of x being -1 and ∞ respectively.

NOTE. It will be noted that $\frac{4}{9}$ is the maximum value and 1 the minimum value, though $\frac{4}{9}$ is less than 1. Thus, *maximum and minimum values do not necessarily correspond to the arithmetically greatest and least values respectively.*

Example 4. *If x be real, prove that the expression* $\dfrac{x^2-x+1}{x^2+x+1}$ *lies between 3 and $\frac{1}{3}$.*

Let $y=\dfrac{x^2-x+1}{x^2+x+1}$; then $(y-1)x^2+(y+1)x+(y-1)=0$.

$$\therefore \quad x=\frac{-(y+1)\pm\sqrt{(y+1)^2-4(y-1)^2}}{2(y-1)}.$$

The expression under the radical sign

$$\begin{aligned}
&=(y+1)^2-4(y-1)^2\\
&=(y^2+2y+1)-4(y^2-2y+1)\\
&=-3y^2+10y-3\\
&=-(y-3)(3y-1)\\
&=-3(y-\tfrac{1}{3})(y-3).
\end{aligned}$$

In order that x may be real, $-3(y-\frac{1}{3})(y-3)$ must be either zero or positive. When it is zero, we have $y=\frac{1}{3}$ or 3; when it is to be positive, one of the factors $y-\frac{1}{3}$ and $y-3$ must be positive and the other negative. Now, for any value of y lying between $\frac{1}{3}$ and 3, $y-\frac{1}{3}$ is positive and $y-3$ negative, while for any value of y less than $\frac{1}{3}$ both the factors $y-\frac{1}{3}$ and $y-3$ are negative, and for any value of y greater than 3, both of them are positive. Hence y lies between $\frac{1}{3}$ and 3.

Hence for real values of x the values of the given expression range between $\frac{1}{3}$ and 3. Thus $\frac{1}{3}$ is the minimum value and 3 the maximum value of the given expression, corresponding to the values 1 and -1 respectively of x.

Example 5. *Prove that for real values of x, the expression* $\frac{2x^2+x-3}{3x+1}$ *is capable of assuming all real values.*

Let $y=\frac{2x^2+x-3}{3x+1}$; then $2x^2+(1-3y)x-(y+3)=0$.

$$\therefore \quad x=\frac{3y-1\pm\sqrt{(1-3y)^2+8(y+3)}}{4}.$$

The expression under the radical sign

$$=(1-3y)^2+8(y+3)=9y^2+2y+25=(3y+\tfrac{1}{3})^2+\tfrac{224}{9},$$

which is evidently positive for all real values of y. Therefore x will be real for all real values of y. Hence y, can have any real value whatever. Since the expression is thus capable of assuming all real values, it has neither a maximum nor a minimum value.

Example 6. *If the equation* $x^2+6xy+y^2-26x-22y+89=0$ *is to be satisfied by real values of x and, y prove that x cannot lie between* 1 *and* 4, *and y cannot lie between* 2 *and* 5.

(i) Writing the given equation in the form of a quadratic in y, we have $y^2+2(3x-11)y+(x^2-26x+89)=0$

$$\therefore \quad y=-(3x-11)\pm\sqrt{(3x-11)^2-(x^2-26x+89)}.$$

Now the expression under the radical sign

$$\begin{aligned}&=(3x-11)^2-(x^2-26x+89)\\&=8x^2-40x+32\\&=8(x-1)(x-4).\end{aligned}$$

Since y is real, $8(x-1)(x-4)$ must be either zero or positive. It is equal to zero when $x=1$ or 4; it will be positive if the factors $x-1$ and $x-4$ are both positive or both negative, and this is always possible except when x lies between 1 and 4. Hence x cannot lie between 1 and 4.

(ii) Again, writing the given equation in the form of a quadratic in x, we have $x^2+2(3y-13)x+(y^2-22y+89)=0$

$$\therefore \quad x=-(3y-13)\pm\sqrt{(3y-13)^2-(y^2-22y+89)}.$$

Now the expression under the radical sign

$$\begin{aligned}&=(3y-13)^2-(y^2-22y+89)\\&=8y^2-56y+80\\&=8(y-2)(y-5).\end{aligned}$$

Since x is real, $8(y-2)(y-5)$ must be either zero or positive. It is equal to zero when $y=2$ or 5; it will be positive if the factors $y-2$ and $y-5$ are both positive or both negative, and this is always possible except when y lies between 2 and 5. Hence y cannot lie between 2 and 5.

Sign of a Quadratic Expression.

en in Art. 66 that the roots of the quadratic equation $ax^2+bx+c=0$ may be either real or imaginary; and when real, may be either equal or unequal.

The sign of the quadratic expression ax^2+bx+c depends upon the nature of the roots of the quadratic equation $ax^2+bx+c=0$.

(i) If the roots α, β are real and unequal, and if $\alpha>\beta$, then from the relation

$$ax^2+bx+c=a(x-\alpha)(x-\beta)\ldots \qquad \ldots \quad (1)$$

obtained in Art. 72, it is evident that if $x>\alpha$, or if $x<\beta$, then $(x-\alpha)(x-\beta)$ is positive and hence the sign of the expression ax^2+bx+c is the same as the sign of a.
i.e. when x does not lie between α and β

But if $x<\alpha$ and $>\beta$, then $(x-\alpha)(x-\beta)$ is negative so that the sign of the expression is opposite to that of a.
i.e. when $\alpha > x > \beta$ or x lies between α & β

(ii) If the roots are real and equal, say, both equal to α, then the relation becomes

$$ax^2+bx+c=a(x-\alpha)^2 \qquad \ldots \qquad \ldots \quad (2)$$

Since $(x-\alpha)^2$ is always positive for real values of x, the sign of the expression ax^2+bx+c must be the same as the sign of a, for all real values of x.

(iii) If the roots are imaginary, it is expedient to rewrite the expression ax^2+bx+c, thus:

$$ax^2+bx+c=a\left\{\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right\} \qquad \ldots \quad (3)$$

We know from Art. 66 that when the roots are imaginary, $b^2<4ac$, *i.e.*, $4ac-b^2$ is positive, so that the expression within the second brackets on the right-hand side of (3) is always positive; and hence the sign of the expression ax^2+bx+c is the same as that of a, for all real values of x.

All these results may be combined as follows:

The sign of the expression ax^2+bx+c is the same as the sign of a (the co-efficient of x^2) for all real values of x; except only when the roots of the equation $ax^2+bx+c=0$ are real and unequal and the value of x lies between them.

Example 1. *Prove that the expression $6x^2-7x+2$ is positive for all real values of x, except when x lies between $\frac{1}{2}$ and $\frac{2}{3}$.*

We have $6x^2-7x+2=6(x^2-\frac{7}{6}x+\frac{1}{3})=6(x-\frac{1}{2})(x-\frac{2}{3})$.

Now, for any value of x less than $\frac{1}{2}$, both $x-\frac{1}{2}$ and $x-\frac{2}{3}$ are negative, and for any value of x greater than $\frac{2}{3}$, both of them are positive. Hence in either case, their product is positive, and therefore $6x^2-7x+2$ is positive ; while for any value of x lying between $\frac{1}{2}$ and $\frac{2}{3}$, the given expression will be negative, for then one of the factors, *i.e.*, $x-\frac{1}{2}$ will be positive and the other $x-\frac{2}{3}$ negative.

Example 2. *Show that (i) x^2+5x+7 and (ii) $2x^2-8x+8$ are always positive for real values of x.*

(i) Since $x^2+5x+7=(x^2+5x+\frac{25}{4})+\frac{3}{4}=(x+\frac{5}{2})^2+\frac{3}{4}$, the given expression is always positive for real values of x.

(ii) Since $2x^2-8x+8=2(x^2-4x+4)=2(x-2)^2$, the given expression is always positive for real values of x.

These results might also be deduced from Art. 81, for in the first case, the roots of the equation $x^2+5x+7=0$ are imaginary ; and in the second case, the roots of the equation $2x^2-8x+8=0$ are equal. Hence the sign of the expressions is the same as the sign of their first terms, that is to say, positive.

EXERCISE 27.

1. Find, for real values of x, the minimum values of :

(i) $x^2-6x+11$. (ii) $3x^2+11x+7$. (iii) $2x^2-5x+8$.
(iv) $4x^2-7x+3$. (v) $5x^2-10x+18$. (vi) $a^2x^2-b^2$.

2. Find, for real values of x, the maximum values of :

(i) $12x-4x^2-1$. (ii) $3-20x-25x^2$. (iii) $7+5x-2x^2$.
(iv) $-2x^2+5x+6$. (v) $8+5x-3x^2$. (vi) $a^2-b^2x^2$.

3. Prove that the following expressions are positive for all real values of x :

(i) $3x^2-7x+5$. (ii) $2x^2+5x+4$. (iii) $7x^2-3x+1$.
(iv) $(a^2+b^2)x^2-2(a+b)x+2$.

4. Prove that the expression $10x^2-19x+6$ is positive for all real values of x except values lying between $\frac{2}{5}$ and $\frac{3}{2}$.

5. Prove that the expression x^2-4x+1 is always positive, except when x lies between $2+\sqrt{3}$ and $2-\sqrt{3}$.

6. For what values of x will the expression $9x^2-24x+11$ be negative ?

7. Prove that for real values of x, the expression $3x^2-6x+8$ can never be less than 5. [Cal. Int. 1935]

8. If x be real, show that the least value of $4x^2-4x+1$ is 0, and the corresponding value of x is $\frac{1}{2}$. [Cal. Int. 1937]

9. Trace the changes in the sign and magnitude of $-6x^2+x+12$, as x assumes all real values from $-\infty$ to $+\infty$. [All. I.E. 1926]

10. What real value or values should be assigned to p in order that $x^2+2px+1$ may be positive for all real values of x? [All. I.E. 1930]

11. Show that the expression $8x-15-x^2$ can be positive only for values of x which lie between certain limits; and find these limits. [All. I.E. 1930]

12. Find for what relation between the co-efficients and what values of x, the expression m^2x^2+bx+c is negative. [Mad. F.A. 1883]

13. If x is real, prove that

(i) $\dfrac{x+1}{x^2-4}$, (ii) $\dfrac{(x-2)(x-3)}{4x-9}$, (iii) $\dfrac{(x-3)(x-5)}{(x-2)(x-4)}$

can have any real value.

14. If x be real, prove that

(i) $\dfrac{x^2+x+2}{x^2+2x+4}$ lies beween $\frac{1}{2}$ and $\frac{7}{6}$;

(ii) $\dfrac{2x^2-x+8}{3x^2-2x+11}$ lies between $\frac{3}{4}$ and $\frac{21}{32}$;

(iii) $\dfrac{x^2-3x+4}{x^2+3x+4}$ lies between $\frac{1}{7}$ and 7. [Cal. Int. 1940]

15. If x be real, find the limits between which

(i) $\dfrac{5x^2-2x+3}{2x^2-2x+1}$ must lie; and

(ii) $\dfrac{2x^2+3x+2}{x^2+3x+1}$ cannot lie.

16. Show that the expression $\dfrac{1}{x+1}+\dfrac{1}{3x+1}-\dfrac{1}{(x+1)(3x+1)}$ cannot lie between 1 and 4, for real values of x. [Mad. F.A. 1880]

17. If x be real, show that the expression $\dfrac{m^2}{1+x}-\dfrac{n^2}{1-x}$ can have any real value. [Mad. F.A. 1883]

18. Find the least value of $\dfrac{6x^2-22x+21}{5x^2-18x+17}$ for all real values x. [Cal. Int. 1942]

19. If $p > 1$, then for all real values of x, the expression $\dfrac{x^2-2x+p^2}{x^2+2x+p^2}$ lies between $\dfrac{p-1}{p+1}$ and $\dfrac{p+1}{p-1}$.

20. If $p > m$, then for all real values of x, the expression $\dfrac{x^2-2mx+p^2}{x^2+2mx+p^2}$ lies between $\dfrac{p-m}{p+m}$ and $\dfrac{p+m}{p-m}$.

21. Find the values of a which make the expression $x^2-ax+1-2a^2$ always positive for real values of x.

[Bom' P. E. 1892]

22. If x be real, prove that

(i) $\dfrac{3x^2+4x+2}{x^2+4x+2}$ cannot lie between -1 and $+1$.

(ii) $\dfrac{2x-3}{2x^2+6x-1}$ cannot lie between $\frac{1}{11}$ and 1.

(iii) $\dfrac{x^2+34x-71}{x^2+2x-7}$ can have no value between 5 and 9.

[Cal. Int. 1938]

(iv) $\dfrac{3x^2+x+1}{x^2-3x+2}$ cannot have any value lying between $-17-10\sqrt{3}$ and $17+10\sqrt{3}$.

[Mad. F.A. 1897]

23. If x and y be two real quantities satisfying the equation $x^2+6xy+y^2+4x+4y+20=0$, show that neither x nor y can lie between -2 and 1.

24. Find the limits to the real values of x and y which can satisfy the equation $4x^2+2xy+y^2-39x-6y+99=0$.

25. If x and y be two real quantities connected by the equation $x^2-4xy-32y^2-2x+40y-11=0$, show that x is real for all real values of y, and y is real for all real values of x, except when x lies between $2(1-\frac{1}{3}\sqrt{6})$ and $2(1+\frac{1}{3}\sqrt{6})$.

26. If the equation $x^2-2xy+3y^2+4x-12y+10=0$ is to be satisfied by real values of x and y, then x must lie between $-\sqrt{3}$ and $\sqrt{3}$, and y between 1 and 3.

27. Prove that the expression $x^2+2bx+c$ is greater than, equal to or less than $(x+b)^2$, according as the roots of the equation $x^2+2bx+c=0$ are imaginary, equal or real.

[Pun. I.E. 1892]

28. Show that the expression

$$\frac{(x^2-4)(x^2+3x+2)(x^2-x-2)+10}{x^2+5x+7}$$

itive for all real values of x. [Mad. F.A. 1881]

Condition for a Common Root.

Let α be the common root of the two quadratic equations $ax^2+bx+c=0$, and $a'x^2+b'x+c'=0$.

Then, since α satisfies both the equations, we have

$$a\alpha^2+b\alpha+c=0,$$
$$a'\alpha^2+b'\alpha+c'=0.$$

By cross-multiplication, we have

$$\frac{\alpha^2}{bc'-b'c}=\frac{\alpha}{ca'-c'a}=\frac{1}{ab'-a'b} \quad \cdots \quad \cdots \quad (1)$$

$$\therefore \quad \alpha^2=\frac{bc'-b'c}{ab'-a'b} \text{ and } \alpha=\frac{ca'-c'a}{ab'-a'b},$$

$$\frac{bc'-b'c}{ab'-a'b}=\left(\frac{ca'-c'a}{ab'-a'b}\right)^2$$

or, $$(ca'-c'a)^2=(bc'-b'c)(ab'-a'b) \quad \cdots \quad \cdots \quad (2)$$

which is the condition required.

From (1), since $\dfrac{\alpha^2}{bc'-b'c}=\dfrac{\alpha}{ca'-c'a}$ $\therefore$ $\alpha=\dfrac{bc'-b'c}{ca'-c'a}$.

Also, since $\dfrac{\alpha}{ca'-c'a}=\dfrac{1}{ab'-a'b}$ $\therefore$ $\alpha=\dfrac{ca'-c'a}{ab'-a'b}$.

Thus, the common root is $\dfrac{bc'-b'c}{ca'-c'a}$ or $\dfrac{ca'-c'a}{ab'-a'b}$, which are evidently equal, as is clear from (2).

$\therefore$ the remaining root of the first equation is

$$\frac{c}{a}\div\frac{bc'-b'c}{ca'-c'a}, \quad \text{or} \quad \frac{c}{a}\div\frac{ca'-c'a}{ab'-a'b}$$

i.e., $$\frac{c(ca'-c'a)}{a(bc'-b'c)}, \quad \text{or} \quad \frac{c(ab'-a'b)}{a(ca'-c'a)};$$

and the remaining root of the second equation is

$$\frac{c'}{a'}\div\frac{bc'-b'c}{ca'-c'a}, \quad \text{or} \quad \frac{c'}{a'}\div\frac{ca'-c'a}{ab'-a'b}$$

i.e., $$\frac{c'(ca'-c'a)}{a'(bc'-b'c)}, \quad \text{or} \quad \frac{c'(ab'-a'b)}{a'(ca'-c'a)}.$$

Cor. *Condition for a common linear factor of the expressions ax^2+bx+c and $a'x^2+b'x+c'$.*

Since a factor of the expression ax^2+bx+c corresponds to a root of the equation $ax^2+bx+c=0$, it follows that if the two given expressions have a common factor, the corresponding equations will have a common root. Hence the condition required is the same as that obtained above for a common root.

83. Condition for a common linear factor.

If the expressions $ax^2+2hxy+by^2$ and $a'x^2+2h'xy+b'y^2$ have a common factor, then the equations

$$ax^2+2hxy+by^2=0 \text{ and } a'x^2+2h'xy+b'y^2=0,$$

$$a\left(\frac{x}{y}\right)^2+2h\left(\frac{x}{y}\right)+b=0, \text{ and } a'\left(\frac{x}{y}\right)^2+2h'\left(\frac{x}{y}\right)+b'=0,$$

which are quadratic equations in $\frac{x}{y}$, must have a common root.

Hence it follows from Art. 82 that the condition is

$$(a'b-ab')^2=4(ah'-a'h)(b'h-bh').$$

84. Condition for rational factors of ax^2+bx+c.

If the expression ax^2+bx+c is to have rational factors, the equation $ax^2+bx+c=0$ must have rational roots.

Hence, from Art. 66, the required condition is that b^2-4ac *must be either zero or a perfect square.*

Further, if b^2-4ac is zero, the expression is a *perfect square.* [See Art. 72, Cor. 2.]

Cor. It follows that $ax^2+2hxy+by^2$ will have rational factors if h^2-ab is a perfect square, and will be a perfect square if $h^2=ab$.

85. Condition for linear factors of the expression $ax^2+2hxy+by^2+2gx+2fy+c$.

Arranging in descending powers of x, we have the given expression $=ax^2+2(hy+g)x+(by^2+2fy+c)$.

Hence considering the expression as a quadratic in x, it can be resolved into linear rational factors.

if $\quad 4(hy+g)^2-4a(by^2+2fy+c)$ is a perfect square,

i.e., if $(h^2-ab)y^2+2(hg-af)y+(g^2-ac)$ is a perfect square; and from Art. 84 the condition for that is

$$4(hg-af)^2-4(h^2-ab)(g^2-ac)=0$$

i.e., $\quad 4a(abc+2fgh-af^2-bg^2-ch^2)=0.$

Hence, since a is not zero, the required condition is

$$abc+2fgh-af^2-bg^2-ch^2=0.$$

Example 1. *Prove that if the equations* $x^2+bx+ca=0$, *and* $x^2+cx+ab=0$ *have a common root, their other roots will satisfy the equation* $x^2+ax+bc=0$. [*Cal. F.A.* 1892]

Let α be the common root; then we have

$$\alpha^2+b\alpha+ca=0 \quad \cdots \quad \cdots \quad (1)$$

and

$$\alpha^2+c\alpha+ab=0 \quad \cdots \quad \cdots \quad (2)$$

Subtracting (2) from (1)

$$(b-c)\alpha+a(c-b)=0$$

$\therefore$ dividing by $b-c$, $\alpha=a$.

Substituting this value of α in (1), we have

$$a^2+ba+ca=0$$

or,

$$a(a+b+c)=0$$

$$\therefore \quad a+b+c=0 \quad \cdots \quad \cdots \quad (3)$$

Since $\alpha=a$ is the common root of $x^2+bx+ca=0$, and $x^2+cx+ab=0$ and since the product of the roots of the first equation $=ca$, and the product of the roots of the second equation $=ab$, the other two roots of these equations are c and b respectively. And these roots will satisfy the equation

$$x^2-(b+c)x+bc=0.$$

Hence on account of the relation (3), they will satisfy the equation $x^2+ax+bc=0$.

Example 2. *If the equations* $ax^3+3bx^2+3cx+d=0$ *and* $ax^2+2bx+c=0$ *have a common root, prove that* $(bc-ad)^2=4(ac-b^2)(bd-c^2)$. [*Cal. F. A.* 1889]

Let α be the commen root, then we have

$$a\alpha^3+3b\alpha^2+3c\alpha+d=0 \quad \cdots \quad \cdots \quad (1)$$

and

$$a\alpha^2+2b\alpha+c=0 \quad \cdots \quad \cdots \quad (2)$$

Multiplying (2) by α and subtracting from (1),

$$b\alpha^2+2c\alpha+d=0 \quad \cdots \quad \cdots \quad (3)$$

From (2) and (3), by the rule of cross multiplication,

$$\frac{\alpha^2}{2(bd-c^2)}=\frac{\alpha}{bc-ad}=\frac{1}{2(ac-b^2)}$$

Hence eliminating α, we have

$$(bc-ad)^2=4(ac-b^2)(bd-c^2).$$

EXERCISE 28.

1. Find the condition that the two quadratic equations $ax^2+bx+c=0$ and $a'x^2+b'x+c'=0$ may have two roots in common.

2. Find the condition that a root of the equation $ax^2+bx+c=0$ may be the reciprocal of a root of the equation $a'x^2+b'x+c'=0$.

3. If the equations $x^2+px+q=0$ and $x^2+p'x+q'=0$ have a common root, show that it must be either

$$\frac{pq'-p'q}{q-q'} \text{ or } \frac{q-q'}{p'-p}.$$ [Cal. Int. 1911]

4. If $x^2+px+q=0$ and $x^2+qx+p=0$ have a common root, show that either $p=q$, or $p+q+1=0$. [Cal. Int. 1939]

5. If the equations $ax^2+bx+c=0$ and $bx^2+cx+a=0$ have a common root, then either $a=0$ or $a^3+b^3+c^3=3abc$.

6. If $x-a$ is a factor of $a_1x^2+2b_1x+c_1$ and $x+a$ of $a_2x^2+2b_2x+c_2$, prove that

$$(c_2a_1-c_1a_2)^2+4(a_1b_2+a_2b_1)(b_1c_2+b_2c_1)=0.$$

[Mad. F.A. 1890]

7. Show that the equations $(b-c)x^2+(c-a)x+(a-b)=0$ and $(c-a)x^2+(a-b)x+(b-c)=0$ have a common root.

8. If the equations $x^2+abx+c=0$ and $x^2+acx+b=0$ have a common root, show that their other roots satisfy the equation $a(b+c)x^2+(b+c)x-abc=0$.

9. If y and z be the roots of the equation $A(x^2+m^2)+Amx+Cm^2x^2=0$, show that $A(y^2+z^2)+Ayz+Cy^2z^2=0$.

[Bom. 1891]

10. If α, β and γ, δ be the roots of $x^2-px+q=0$, and $x^2-p'x+q'=0$ respectively, find the value of

$$(\alpha-\gamma)(\beta-\delta)+(\beta-\gamma)(\alpha-\delta).$$ [Cal. Int. 1928]

11. What values of m will make $6x^2-7xy-3y^2+mx+17y-20$ resolvable into rational factors? Find the corresponding factors in each case.

12. Show that if $ax^2+by^2+cz^2+2ayz+2bzx+2cxy$ is resolvable into rational factors, $a^3+b^3+c^3=3abc$.

13. What must be added to $a^2x^2+2bxy+c^2y^2$ in order to make it a perfect square? [Pun. I.E. 1889]

14. For what values of x will both the expressions $x^2-(2a-b)x+a^2-ab$ and $x^2-(2a+b)x+a^2+ab$ vanish?

15. Determine r so that the equation $\dfrac{2p}{x+a}+\dfrac{r}{x}+\dfrac{2q}{x-a}=0$ may have equal roots; and if r_1, r_2 be the two values of r, and x_1, x_2 the corresponding values of x, show that $x_1x_2=a^2$ and $r_1r_2=(p-q)^2$.

16. Find the conditions that the roots of $a'x^2+b'x+c'=0$ may be respectively the squares of the roots of $ax^2+bx+c=0$.

17. If α and β are the roots of $x^2+px+q=0$, show that p and q are the roots of the equation $x^2+(\alpha+\beta-\alpha\beta)x-\alpha\beta(\alpha+\beta)=0$.

18. If $\dfrac{ax-b}{1}=\dfrac{bx-c}{y}=\dfrac{cx+a-1}{y^2}$, and if b^2-ac is not zero, prove that x and y are the roots of a certain quadratic equation.

19. Determine the condition that two of the roots of the equation $ax^3+bx^2+cx+d=0$ may be the roots of the quadratic equation $px^2+qx+r=0$.

20. Prove that if x is real, the expression $\dfrac{(x-a)(x-c)}{x-b}$ is capable of assuming all values, if a, b, c are in ascending order of magnitude. [Mad. F.A. 1889]

21. Prove that the expression $\dfrac{x+a}{x^2+bx+c^2}$ will always lie between two fixed limits if $b^2<4c^2$; there will be two limits between which it cannot lie if $a^2+c^2>ab$ and $b^2>4c^2$; and the expression is capable of all values if $a^2+c^2<ab$.

22. Find the condition that the roots of the equation $\dfrac{x+a}{a}\cdot\dfrac{x+b}{b}\cdot\dfrac{x+c}{c}=1$ may be possible. [Mad. F.A. 1882]

23. Show that if x is real, $\dfrac{(x-a)(x-b)}{(x-c)(x-d)}$ can have any value provided that only one of the two quantities a and b lies between c and d.

24. Show that for real values of x, $\dfrac{ax^2+2bx+c}{cx^2+2bx+a}$ is unlimited in value if $a+c<2b$.

CHAPTER IX

SIMULTANEOUS EQUATIONS INVOLVING QUADRATICS.

86. Number of Equations and Solutions.

In a system of simultaneous equations *there must be as many equations as there are unknown quantities to be determined.*

Thus if the unknown quantities are two in number, x and y, the number of equations necessary to determine them must also be two; if the unknown quantities are three in number, x, y and z, the number of equations must be three also; and so on. All these equations must be *independent*; that is to say, not derivable from one or more of the others. Further they must be *consistent* with one another.

Again, *the number of sets of solutions in a system of simultaneous equations is always equal to the product of the degrees of the equations.*

Thus, if there are two simultaneous equations involving two unknown quantities, and one of the equations is of the first degree (or linear) and another of the second degree (or quadratic), there will be 1×2 or 2 sets of solutions; if one equation is a quadratic and the other a cubic (*i.e.,* of the third degree), there will be 2×3 or 6 sets of solutions. Similarly, if there be three simultaneous equations involving three unknown quantities, and one equation is a quadratic, another is a cubic and the third one a quartic or biquadratic (*i.e.*, of the fourth degree) then the number of sets of solutions will be $2 \times 3 \times 4$ or 24. The total number thus arrived at will, of course, include imaginary as well as real, infinite as well as finite solutions.

87. We shall in this chapter deal with some types of simultaneous equations involving two or three unknowns, where one or more of the equations may be of a degree higher than the first, but the solution of which ultimately depends upon that of a quadratic or some equation reducible to a quadratic. Thus some groups of equations of a degree higher than the quadratic will also be treated, when their solutions can be effected in this manner.

I. Equations involving two unknowns

88. Methods of Solution.

There is no general rule for the solution of a pair of simultaneous equations involving two unknown quantities x and y, beyond this that if we can *eliminate* any one of the unknown quantities between the two equations we shall obtain an equation in the other unknown, and if such equation turns out to be a quadratic or an equation reducible to a quadratic, the latter unknown can be found and hence the other.

There are however some distinct type of equations to which particular methods are suited; and we proceed to discuss them.

89. One equation Linear and the other Quadratic: Method of Substitution.

When one of the simultaneous equations is linear and the other a quadratic, the usual method of solution is to express one of the unknown quantities in terms of the other by means of the linear equation and then *substitute* for it the value so found in the other equation. Then the resulting equation will be a quadratic which can be solved in the usual manner.

Example 1. *Solve*: $\left.\begin{array}{rl} 5x+2y=12 & (1) \\ 2x^2+3xy+y^2=15 & (2) \end{array}\right\}$ [*Cal. F.A.* 1888]

From (1) we have $\quad x=\dfrac{12-2y}{5} \quad \cdots \quad \cdots \quad (3)$

Substituting this value of x in (2) we have

$$2\left(\frac{12-2y}{5}\right)^2+3y\left(\frac{12-2y}{5}\right)+y^2=15$$

or. $\quad 2(12-2y)^2+15y(12-2y)+25y^2=375$

or, $\quad 3y^2+84y-87=0$

or, $\quad y^2+28y-29=0$

or, $\quad (y-1)(y+29)=0$

$\therefore \quad y=1$ or -29.

Hence from (3), when $y=1$, $x=\dfrac{12-2\times 1}{5}=2$;

and when $y=-29$, $x=\dfrac{12-2\times(-29)}{5}=14$.

Thus the required solutions are $x=2$, $y=1$; and $x=14$, $y=-29$.

NOTE 1. Since the degree of the first equation is 1 and that of the second is 2, the number of solutions is 1×2, *i.e.*. 2, and we have got accordingly 2 sets of values of x and y.

NOTE 2. The student must carefully note the corresponding values of x and y and arrange them in sets correctly.

Thus, in the above example, the set $x=2$ and $y=-29$ or the set $x=14$ and $y=1$, do not satisfy the given equations, and therefore neither of them form the corresponding values of x and y.

Example 2. *Solve* : $\left.\begin{aligned}\frac{1}{x}+\frac{1}{y}&=\frac{1}{2} &\cdots(1)\\ x+y&=9 &\cdots(2)\end{aligned}\right\}$ [*Cal. F.A.* 1906]

From (1), we have $2(x+y)=xy$ ... ... (3)

From (2), we have $x=9-y$ ... ... (4)

Substituting this value of x in (3),

$$2\times 9=y(9-y)$$

or, $y^2-9y+18=0$

or, $(y-3)(y-6)=0$

$\therefore$ $y=3$, or 6.

Hence from (4), when $y=3$, $x=9-3=6$,

and when $y=6$, $x=9-6=3$.

Thus, the required solutions are $x=6, y=3$ and $x=3, y=6$.

90. Special Artifices.

Sometimes special artifices other than the standard method laid down in the previous article may be usefully employed to effect the solution.

If instead of x and y, their reciprocals $\frac{1}{x}$ and $\frac{1}{y}$ occur throughout the equations, the same methods will do.

Example 1. *Solve* : $\left.\begin{aligned}x+y&=24 &\cdots(1)\\ xy&=63 &\cdots(2)\end{aligned}\right\}$

Squaring (1), $x^2+y^2+2xy=576$

From (2), $4xy=252$

Hence by subtraction, $x^2+y^2-2xy=324$

Taking the square root, $x-y=\pm 18$ ... (3)

Thus we get two sets of linear simultaneous equations from which to determine x and y; *viz.*

$$\left.\begin{array}{l} x+y=24 \\ x-y=18 \end{array}\right\} \quad \text{and} \quad \left.\begin{array}{l} x+y=24 \\ x-y=-18 \end{array}\right\}$$

From the first set. we get $x=21$, $y=3$.
From the second set, we get $x=3$, $y=21$.

Hence, the required two sets of solutions are $x=21$, $y=3$; and $x=3$, $y=21$.

NOTE. The pair of equations of the form

$$\left.\begin{array}{r} x-y=a \\ xy=b \end{array}\right\}$$

are solved in a similar manner. The identity used in both these cases is $(x+y)^2-(x-y)^2=4xy$.

Example 2. *Solve* :

$$\left.\begin{array}{rlr} x+y=7 & \cdots \quad \cdots & (1) \\ x^2+y^2=37 & \cdots \quad \cdots & (2) \end{array}\right\}$$

Squaring (1), $x^2+y^2+2xy=49$
Subtracting (2), $2xy=12$
or, $xy=6 \quad \cdots \quad \cdots \quad (3)$

The solution is thus reduced to that of equations (1) and (3), which can be effected as in Example 1. The solutions will be found to be : $x=6$, $y=1$; and $x=1$, $y=6$.

NOTE. The pair of equations of the form

$$\left.\begin{array}{r} x-y=a \\ x^2+y^2=b \end{array}\right\}$$

are solved in a similar manner.

Example 3. *Solve* :

$$\left.\begin{array}{rlr} x-y=6 & \cdots \quad \cdots & (1) \\ x^2-y^2=48 & \cdots \quad \cdots & (2) \end{array}\right\}$$

Dividing equation (2) by equation (1), we have

$$x+y=8 \quad \cdots \quad \cdots \quad (3)$$

The solution is thus reduced to that of the linear equations (1) and (3). The solution evidently is : $x=7$, $y=1$.

NOTE 1. We would expect here two solutions, as one equation is linear and the other quadratic. In fact, there are two solutions, one solution leading however to *infinite* values of x and y.

Both the solutions will come out if we adopt the standard *method of substitution* Thus,

From (1), we have $x=y+6 \quad \cdots \quad \cdots \quad (4)$
Substituting in (2), $(y+6)^2-y^2=48$
or, $0.y^2+12y-12=0.$

The co-efficient of y^2 in this quadratic equation vanishes. Hence one root of the equation is *infinite* (Art. 70]; the other root is of course finite and equal to 1 as already found.

From (4), when $y=\infty$, $x=\infty$; and when $y=1$, $x=7$.

Thus the two solutions are : $x=\infty$, $y=\infty$; and $x=7$, $y=1$.

NOTE 2. The pair of equations of the form

$$\left.\begin{aligned} x+y&=a \\ x^2-y^2&=b \end{aligned}\right\}$$

are solved in a similar manner.

91. Some simpler types of simultaneous equations in which both the equations are quadratic, or in which one equation is of a higher degree and the other linear, can also be easily solved by similar artifices.

Example 1. *Solve* :

$$\left.\begin{aligned} x^2+y^2&=74 \quad \ldots \quad \cdots (1) \\ xy&=35 \quad \ldots \quad \cdots (2) \end{aligned}\right\}$$

From the given equations we have

$$x^2+y^2+2xy=74+2\times 35=144$$

and $$x^2+y^2-2xy=74-2\times 35=4$$

Hence extracting the square roots,

$$x+y=\pm 12, \text{ and } x-y=\pm 2.$$

Thus we have 4 sets of linear simultaneous equations, *viz.*

$$\left.\begin{aligned} x+y&=12 \\ x-y&=2 \end{aligned}\right\} \quad \cdots \quad (A) \qquad \left.\begin{aligned} x+y&=12 \\ x-y&=-2 \end{aligned}\right\} \quad \cdots (B)$$

$$\left.\begin{aligned} x+y&=-12 \\ x-y&=2 \end{aligned}\right\} \quad \cdots \quad (C) \qquad \left.\begin{aligned} x+y&=-12 \\ x-y&=-2 \end{aligned}\right\} \quad \cdots (D)$$

By addition and subtraction we have the following solutions :

$$\left.\begin{aligned} x&=7 \\ y&=5 \end{aligned}\right\} \quad \begin{aligned} x&=5 \\ y&=7 \end{aligned} \quad \left.\begin{aligned} x&=-5 \\ y&=-7 \end{aligned}\right\} \quad \left.\begin{aligned} x&=-7 \\ y&=-5 \end{aligned}\right\}$$

from the equations (A), (B), (C) and (D) respectively.

NOTE. Since the degree of the first equation is 2 and that of the second is also 2, the number of solutions is 2×2, *i.e.*, 4.

Example 2. *Solve* :

$$\left.\begin{aligned} x+\frac{1}{y}&=a \quad \cdots \quad (1) \\ y+\frac{1}{x}&=b \quad \cdots \quad (2) \end{aligned}\right\} \qquad [\textit{Cal. F.A. 1884}]$$

From (1), we have $xy+1=ay$ ... ... (3)

and from (2), we have $xy+1=bx$ ... ... (4)

$$\therefore \qquad ay=bx, \text{ or, } y=\frac{b}{a}x.$$

Substituting this value of y in either (3) or (4), we have

$$\frac{b}{a}x^2+1=bx,$$

or, $$bx^2-abx+a=0$$

$$\therefore \qquad x=\frac{ab\pm\sqrt{(a^2b^2-4ab)}}{2b}$$

and $$y=\frac{b}{a}x=\frac{ab\pm\sqrt{(a^2b^2-4ab)}}{2a},$$

which are the required solutions.

Example 3. *Solve :* $\left.\begin{array}{r} x^3+y^3=9 \quad (1) \\ x+y=3 \quad (2) \end{array}\right\}$ [*Cal. Int.* 1916]

The equation (1) may be written

$$(x+y)^3-3xy(x+y)=9$$

Hence, from (2), $3^3-3xy\,3=9$

or, $9xy=18$

or, $xy=2$ ... ... (3)

The solution is thus reduced to the solution of equations (2) and (3), which may be effected as in Example 1, Art. 90.

The solutions are : $x=2$, $y=1$; and $x=1$, $y=2$.

NOTE. In solving by this method we have got only two sets of solutions, whereas the total number should be three. In fact, there are three solutions, the third solution giving however *infinite* values of x and y. [Compare Example 3, Art. 90.]

All the three solutions will come out if we adopt the standard *method of solution.* Thus :

From (2), we have $y=3-x$

Substituting in (1), $x^3+(3-x)^3=9$

or, $0.x^3+9x^2-27x+18=0$ ... ... (4)

The co-efficient of x^3, the highest power of x in the equation, being zero, one root is infinite [Art. 70]; and the other two roots are 2 and 1 as before. The value of $y=-\infty$ corresponds to the value $x=\infty$.

92. As is suggested by the artifices in the previous articles, we see that in many cases, in order to solve the given equations, what we aim at is to find out the values of one of the following pairs of quantities, which when obtained, lead easily to the valuos of x and y:

(i) $x+y$ and $x-y$; (ii) x^2+y^2 and x^2-y^2;
(iii) $x+y$ and xy; (iv) $x-y$ and xy;
(v) x^2+y^2 and xy; (vi) x^2-y^2 and xy;
(vii) x^2+y^2 and $x+y$; (viii) x^2+y^2 and $x-y$;
(ix) x^2-y^2 and $x+y$; (x) x^2-y^2 and $x-y$.

EXERCISE 29.

Solve the following equations:

1. $x^2-xy+6y^2=27,\ x+y=5.$

2. $3x^2-5xy+4y^2=9,\ 7x-3y=1.$

3. $2x^2+30xy+25y^2=21,\ 5(x-2y)=8.$

4. $2x^2+3xy+4y^2=24,\ x+3y=7.$ [Cal. Int. 1916]

5. $x+y=3,\ 2x^2-5xy+2y^2=0.$ [Cal. Int. 1920]

6. $x+y=20,\ xy=91.$ **7.** $x-y=3,\ xy=40.$

8. $x^2+y^2=50,\ x+y=8.$ **9.** $x^2+y^2=394,\ x-y=2.$

10. $x-y=5,\ x^2-y^2=45.$ **11.** $x+y=11,\ x^2-y^2=99.$

12. $x-y=2,\ xy=-2,$ [Cal. Int. 1915]

13. $5x+3y=39,\ xy=24.$

14. $x^2+xy+y^2=109,\ x+y=12.$

15. $\dfrac{2}{x}+\dfrac{3}{y}=2,\ 5x-3y=1.$

16. $\dfrac{1}{x-3}+\dfrac{1}{y-2}=5,\ 2x+3y=14.$

17. $\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{1}{2},\ x+y=9.$ [Cal. F.A. 1906]

18. $x+\dfrac{4}{y}=1,\ y+\dfrac{4}{x}=25.$ [Cal. Int. 1920]

19. $x+\dfrac{3}{y}=2,\ y+\dfrac{3}{x}=-2.$ [Cal. F.A. 1879]

20. $\dfrac{a}{x}+\dfrac{b}{y}=2,\ \dfrac{a^2}{x^2}+\dfrac{b^2}{y^2}=2.$ [Cal. Int. 1909 & 1910]

21. $\frac{1}{x}-\frac{1}{y}=1,\ \frac{1}{x^2}+\frac{1}{y^2}=13.$

22. $\frac{a^2}{x}-\frac{b^2}{y}=\frac{(a-b)^2}{c}$; $x-y=c.$ [Mad. F.A. 1893]

23. $3x+2y=2xy,\ 9x+4y=5xy.$ [Cal. F.A. 1895]

24. $\frac{x}{y}+\frac{y}{x}=\frac{58}{21}$; $x+y=10.$

25. $2\sqrt{\frac{x}{y}}+3\sqrt{\frac{y}{x}}=5,\ x+y=13.$

26. $x+\frac{1}{y-2}=3,\ y+\frac{1}{x-1}=4.$

27. $ax-by=a^2-b^2,\ \frac{a}{x+a}+\frac{b}{y+b}=\frac{1}{2}.$

28. $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{10}{3}$; $x+y=10.$ [Cal. Int. 1918]

29. $x^2+xy=15,\ x-y=1.$ [Cal. Int. 1915]

30. $x+y=6,\ x^3+y^3=72.$ [Cal. F.A. 1908]

31. $x+y=a,\ x^3+y^3=b^3.$ [Cal. Int. 1912]

32. $x^3-y^3=218,\ x-y=2.$ [Cal. Int. 1917]

33. $x+xy=3,\ y+xy=4.$ [Cal. Int. 1921]

34. $x^2+xy+y^2=93,\ x^2-xy+y^2=37.$

35. $x^2+xy+y^2=189;\ x+\sqrt{xy}+y=21.$

36. $\frac{1}{x^3}-\frac{1}{y^3}=\frac{1}{b}\left(\frac{3}{a^2}+\frac{1}{b^2}\right)$, $\frac{1}{x}-\frac{1}{y}=\frac{1}{b}.$ [Cal. F.A. 1891]

93. Homogeneous Equations.

When the terms involving x and y in both the equations are homogenous and of the same degree, then such pairs of equations are solved by either of the methods illustrated in the examples given below.

Example 1. *Solve* :
$$\left.\begin{aligned} x^2-3xy+2y^2&=2 \quad \cdots \quad \cdots \quad \cdots (1)\\ 4xy-12y^2-x^2&=-9 \quad \cdots \quad \cdots \quad \cdots (2)\end{aligned}\right\}$$

First Method : Dividing (1) by (2), we have

$$\frac{x^2-3xy+2y^2}{4xy-12y^2-x^2}=-\frac{2}{9}$$

or, $\quad 9(x^2-3xy+2y^2)=-2(4xy-12y^2-x^2) \quad \cdots \quad (3)$

or, $\quad 7x^2-19xy-6y^2=0 \quad \cdots \quad \cdots \quad (4)$

or, $\quad (x-3y)(7x+2y)=0$

$\therefore \quad x=3y, \text{ or } -\frac{2}{7}y \quad \cdots \quad \cdots \quad (5)$

Taking the first value, and substituting in either of the given equations, say (1), we have

$$(3y)^2-3.(3y).y+2y^2=2, \quad \text{or} \quad 2y^2=2 \quad \text{or } y^2=1.$$

$\therefore \quad y=\pm 1$, and hence $x=\pm 3$.

Taking the second value, and substituting in (1), we have

$$(-\tfrac{2}{7}y)^2-3(-\tfrac{2}{7}y)y+2y^2=2, \text{ or } \tfrac{144}{49}y^2=2, \text{ or } y^2=\tfrac{98}{144}.$$

$\therefore \quad y=\pm\frac{7}{12}\sqrt{2}$ and hence $x=-\frac{2}{7}y=\mp\frac{1}{6}\sqrt{2}$.

Therefore the four sets of solutions are :

$$\left.\begin{array}{l} x=3 \\ y=1 \end{array}\right\} \quad \left.\begin{array}{l} x=-3 \\ y=-1 \end{array}\right\} \quad \left.\begin{array}{l} x=\frac{1}{6}\sqrt{2} \\ y=-\frac{7}{12}\sqrt{2} \end{array}\right\} \quad \left.\begin{array}{l} x=-\frac{1}{6}\sqrt{2} \\ y=\frac{7}{12}\sqrt{2} \end{array}\right\}$$

Second Method : Putting $y=vx$, the equations become

$$x^2-3vx^2+2v^2x^2=2, \quad \text{or,} \quad (1-3v+2v^2)x^2=2 \quad (6)$$

$$4vx^2-12v^2x^2-x^2=-9, \quad \text{or,} \quad (4v-12v^2-1)x^2=-9 \quad (7)$$

From (6), we have $\quad x^2=\dfrac{2}{1-3v+2v^2} \quad \cdots \quad \cdots \quad \cdots \quad (8)$

From (7), we have $\quad x^2=-\dfrac{9}{4v-12v^2-1} \quad \cdots \quad \cdots \quad (9)$

Equating these values, $\dfrac{2}{1-3v+2v^2}=-\dfrac{9}{4v-12v^2-1}$

or, $\quad 2(4v-12v^2-1)=-9(1-3v+2v^2) \quad \cdots \quad (10)$

or, $\quad 6v^2+19v-7=0$

or, $\quad (3v-1)(2v+7)=0$

or, $\quad v=\frac{1}{3}, \text{ or } -\frac{7}{2} \quad \cdots \quad (11)$

Substituting the first value in (7) or (8), we have

$x^2=9$, and hence $x=\pm 3$. $\quad \therefore y=\frac{1}{3}x=\pm 1$.

Substituting the second value in (7) or (8), we have

$x^2=\frac{1}{18}$, and hence $x=\pm\frac{1}{6}\sqrt{2}$. $\quad \therefore y=-\frac{7}{2}x=\mp\frac{7}{12}\sqrt{2}$.

Hence the solutions are :

$$\left.\begin{array}{l} x=3 \\ y=1 \end{array}\right\} \quad \left.\begin{array}{l} x=-3 \\ y=-1 \end{array}\right\} \quad \left.\begin{array}{l} x=\frac{1}{6}\sqrt{2} \\ y=-\frac{7}{12}\sqrt{2} \end{array}\right\} \quad \left.\begin{array}{l} x=-\frac{1}{6}\sqrt{2} \\ y=\frac{7}{12}\sqrt{2} \end{array}\right\}$$

94. Peculiar Case : Common Factor.

A peculiar case arises if the homogeneous expressions in x and y in the two equations have a *common linear factor.* Then if that linear factor is put equal to zero, and the corresponding relation between x and y obtained, the equation corresponding to (3) above will be satisfied, for both sides will vanish, but this relation will not satisfy the given equations unless both x and y are infinite. In such a case, if we adopt the second method of solution, the equation corresponding to (10) will have a common linear factor on either side, and the values of x given by (8) or (9) corresponding to the value of v found by equating this common factor to zero will be of course infinite, for the denominators will vanish for that value of v. Hence the values of y will be infinite also.

This peculiar case therefore has *two infinite solutions,* $x = \pm\infty$, and the corresponding infinite values (plus or minus) of y. The other two solutions are finite.

Of course, homogeneous expressions in the two equations cannot have both the factors common, without the equations either ceasing to be independent or consistent in which cases there can be no definite solution.

Example 1. *Solve* : $\left.\begin{aligned} x^2+xy-6y^2&=24 \quad \cdots \quad \cdots \quad (1)\\ x^2+3xy-10y^2&=32 \quad \cdots \quad \cdots \quad (2)\end{aligned}\right\}$

Putting $y=vx$, we have

$$(1+v-6v^2)x^2=24$$
$$(1+3v-10v^2)x^2=32$$

or, $$x^2=\frac{24}{1+v-6v^2}=\frac{32}{1+3v-10v^2} \quad \cdots \quad (3)$$

$\therefore$ $\quad 24(1+3v-10v^2)=32(1+v-6v^2)$

or, $\quad 24(1-2v)(1+5v)=32(1-2v)(1+3v)$

Hence either $1-2v=0$, and therefore $v=\frac{1}{2}$;

or, $\quad 24(1+5v)=32(1+3v)$, and therefore $v=\frac{1}{3}$.

Taking $v=\frac{1}{2}$, we have from (3), $x^2=\infty$.

$\therefore$ $\quad x=\pm\infty$, and hence $y=\frac{1}{2}x=\pm\infty$.

Taking $v=\frac{1}{3}$, we have from (3), $x^2=36$.

$\therefore$ $\quad x=\pm 6$, and hence $y=\frac{1}{3}x=\pm 2$.

Hence the two finite solutions are :

$x=6$, $y=2$ and $x=-6$, $y=-2$.

95. Equations involving x^2 and y^2 only.

Homogeneous equations in x and y which do not contain product terms like xy but involve x^2 and y^2 only may be solved like ordinary simultaneous equations of the first degree.

Example 1. *Solve* : $5x^2-9y^2=9$ ... ... (1)
$3x^2+2y^2=35$ ... ... (2)

Multiplying (1) by 2, and (2) by 9, and adding,

$$37x^2=333$$

$\therefore$ $x^2=9$, or $x=\pm 3$.

Hence from (1), $y^2=4$, or $y=\pm 2$.

$\therefore$ the four solutions are :

$$\left.\begin{array}{l}x=3\\y=2\end{array}\right\}\quad \left.\begin{array}{l}x=3\\y=-2\end{array}\right\}\quad \left.\begin{array}{l}x=-3\\y=2\end{array}\right\}\quad \left.\begin{array}{l}x=-3\\y=-2\end{array}\right\}.$$

EXERCISE 30.

Solve the following equations :

1. $x^2+2xy=8,\ 3xy-2y^2=4.$
2. $x(x+3y)=10,\ y(8x-y)=15.$
3. $2xy+y^2=16,\ 2x^2-xy=12.$ [All. I.E. 1897]
4. $x^2-2xy=7,\ 2y^2-xy=-3.$ [Mad. F.A. 1885]
5. $x^2+xy=12,\ xy-y^2=2.$ [Cal. F.A. 1869]
6. $x^2+3xy+2y^2=7,\ 2x^2+xy-y^2=9.$
7. $x^2+y^2=13,\ x^2+2xy-y^2=17.$
8. $5xy-2x^2=12,\ 3x^2+xy=18.$ [Mad. F.A. 1886]
9. $3x^2+2y^2=50,\ xy-3y^2=1.$ [Cal. F.A. 1877]
10. $3y^2-8xy+7x^2=18,\ y^2-x^2=16.$
11. $2x^2+3xy=26,\ 3y^2+2xy=39.$
12. $4xy-x^2=15,\ 39y^2-xy=150.$ [Cal. F.A. 1890]
13. $2x^2+3xy+y^2=20,\ 5x^2+4y^2=41.$ [Cal. F.A. 1892]
14. $x^2+xy=a^2,\ x^2-xy=b^2.$ [Cal. F.A. 1867]
15. $2x^2+5xy+12y^2=16,\ 5x^2+2xy+16y^2=26.$
16. $4x^2+2xy+y^2=3,\ x^2+2xy+y^2=7.$ [All. I.E. 1920]
17. $3x^2-4xy+5y^2=33,\ 4x^2-xy=10.$ [Cal. F.A. 1878]
18. $8x^2-3xy+7y^2=42,\ 4x^2-5xy=14.$

19. $\frac{3}{x} - \frac{5}{y} = -2x, \quad \frac{4}{x} - \frac{3}{y} = y.$

20. $x = \frac{a}{x} + \frac{b}{y}, \quad y = \frac{b}{x} + \frac{a}{y}.$

21. $x^2 + xy = a, \ y^2 + xy = b.$

22. $ax^2 + bxy = p, \ by^2 + axy = q.$

23. $2x^2 + 3y^2 = 14, \ 3x^2 + y^2 = 5.$

24. $4x^2 + 7y^2 = 148, \ 12(x^2 + y^2) = 25xy.$ [Pun. I.E. 1887]

25. $a^2y^2 + b^2x^2 = a^2b^2, \ xy = c^2.$

96. Symmetrical Equations.

An equation is said to be symmetrical with respect to x and y if the equation is unaltered when x and y are interchanged.

When both the equations in x and y are symmetrical, they can be easily solved by assuming for the unknown quantities the sum and difference of two other quantities. The method is illustrated in the following examples.

Example 1. *Solve* : $x^3 + y^3 = 91 \quad \ldots (1)$
$x + y = 7 \quad \ldots (2)$

Assume $x = u + v$, and $y = u - v$.

Then $x + y = (u + v) + (u - v) = 2u.$

$\therefore$ from (2), $2u = 7$, $\therefore u = \frac{7}{2} \quad \ldots (3)$

From (1), $(u + v)^3 + (u - v)^3 = 91$, or, $2u^3 + 6uv^2 = 91$

$\therefore$ writing $\frac{7}{2}$ for u, from (3), we have $2(\frac{7}{2})^3 + 6(\frac{7}{2})v^2 = 91$

or, $21v^2 = \frac{21}{4}$, or, $v^2 = \frac{1}{4}$. $\therefore v = \pm\frac{1}{2} \quad \ldots (4)$

Hence from (3) and (4),

$$x = u + v = \tfrac{7}{2} \pm \tfrac{1}{2} = 4 \text{ or } 3,$$

and $$y = u - v = \tfrac{7}{2} \mp \tfrac{1}{2} = 3 \text{ or } 4.$$

Hence the solutions are : $x = 4, y = 3$; and $x = 3, y = 4$.

NOTE. In the above example there are really three solutions, of which the above two are finite and the third solution gives infinite values of x and y. (See Example 3, Art. 91, for another method.)

Example 2. *Solve :* $\left.\begin{array}{ll}\dfrac{x^2}{y}+\dfrac{y^2}{x}=18 & \ldots (1)\\ x+y=12 & \ldots (2)\end{array}\right\}$ [*Cal. Int.* 1919]

Assume $x=u+v$, and $y=u-v$.

Then from (2), $x+y=(u+v)+(u-v)=2u=12 \therefore u=6 \ldots (3)$

From (1), $\dfrac{x^3+y^3}{xy}=18$, or $x^3+y^3=18xy$

$\therefore \quad (u+v)^3+(u-v)^3=18(u+v)(u-v)$

or, $\quad 2u^3+6uv^2=18(u^2-v^2)$

or, $\quad u^3+3uv^2=9(u^2-v^2) \quad \ldots (4)$

Substituting $u=6$ from (3), we have

$$216+18v^2=9(36-v^2)$$

or, $\quad 24+2v^2=36-v^2$

or, $\quad 3v^2=12$

or, $\quad v^2=4$

$\therefore \quad v=\pm 2 \quad \ldots (5)$

Hence from (3) and (4), we have

$$\left.\begin{array}{ll} & x=u+v=6\pm 2=8 \text{ or } 4\\ \text{and} & y=u-v=6\mp 2=4 \text{ or } 8\end{array}\right\}$$

NOTE. Here also, one value of v is *infinite*, since the co-efficient of v^3 in equation (4) is 0 (Art 70). This value of v leads to the solution $x=\infty$, $y=-\infty$. This infinite solution together with the two finite solutions obtained above make up the three solutions required.

97. Some equations though not symmetrical in x and y may be solved by the above method.

Example 1. *Solve :* $\quad x^4+y^4=257 \quad \ldots (1)$

$$x-y=3 \quad \ldots (2)$$

Let $x=u+v$, and $y=u-v$

Then from (2), $\quad (u+v)-(u-v)=2v=3 \therefore v=\frac{3}{2} \quad \ldots (3)$

Also from (1) $\quad (u+v)^4+(u-v)^4=257$

or, $\quad 2(u^4+6u^2v^2+v^4)=257.$

Substituting from (3),

$$2(u^4+6u^2.\tfrac{9}{4}+\tfrac{81}{16})=257$$

or, $\quad 16u^4+216u^2+81=2056$

or, $\quad 16u^4+216u^2-1975=0,$

or, $\quad (4u^2-25)(4u^2+79)=0,$

or, $\quad u^2=\frac{25}{4}$ or $-\frac{79}{4}$.

Hence $\left.\begin{aligned} x &= \pm\tfrac{5}{2}+\tfrac{3}{2}=4 \text{ or } -1 \\ y &= \pm\tfrac{5}{2}-\tfrac{3}{2}=1 \text{ or } -4 \end{aligned}\right\}$ or $\begin{aligned} x &= \pm\tfrac{1}{2}\sqrt{-79}+\tfrac{3}{2} \\ y &= \pm\tfrac{1}{2}\sqrt{-79}-\tfrac{3}{2} \end{aligned}$

Thus the four sets of solutions are :

$$\left.\begin{aligned} x&=4 \\ y&=1 \end{aligned}\right\} \quad \left.\begin{aligned} x&=-1 \\ y&=-4 \end{aligned}\right\} \quad \left.\begin{aligned} x&=\tfrac{1}{2}(3+\sqrt{-79}) \\ y&=\tfrac{1}{2}(-3+\sqrt{-79}) \end{aligned}\right\} \quad \left.\begin{aligned} x&=\tfrac{1}{2}(3-\sqrt{-79}) \\ y&=\tfrac{1}{2}(-3-\sqrt{-79}) \end{aligned}\right\}$$

EXERCISE 31.

Solve the following equations :

1. $x^3+y^3=35,\ x+y=5.$ **2.** $x^3-y^3=485,\ x-y=5.$

3. $x^4+y^4=82,\ x+y=4.$ **4.** $x^4+y^4=1312,\ x+y=8.$

5. $x^4+y^4=2482,\ x-y=1.$ **6.** $x^4+y^4=1552,\ x-y=10.$

7. $x^5+y^5=1056,\ x+y=6.$ **8.** $x^5-y^5=31,\ x-y=1.$

9. $\dfrac{x^2}{y}+\dfrac{y^2}{x}=13\tfrac{1}{2},\ x+y=9.$ **10.** $\dfrac{x^2}{y^2}+\dfrac{y^2}{x^2}=34,\ x+y=1.$

11. $x+y=4,\ (x^2+y^2)(x^3+y^3)=280.$

12. $\dfrac{a}{x}+\dfrac{b}{y}=\dfrac{1}{a}+\dfrac{1}{b},\ \dfrac{x}{a}+\dfrac{y}{b}=a+b.$

13. $\dfrac{a}{2x-a}+\dfrac{b}{2y-b}=2,\ \dfrac{x}{a}+\dfrac{y}{b}=2$

14. $\dfrac{a}{a+x}+\dfrac{b}{b+y}=1,\ x+y=a+b.$

15. $\dfrac{a^2}{x}+\dfrac{b^2}{y}=\dfrac{(a+b)^2}{c},\ x+y=c.$

16. $x^3+y^3=\tfrac{9}{2}xy,\ x+y=3.$ [Cal. Int. 1918]

98. Miscellaneous Equations.

We shall now solve a few simultaneous equations of miscellaneous types by the use of special artifices. They can however be otherwise solved by the standard methods already explained.

Example 1. *Solve* : $\left.\begin{aligned} (1+x)(1+y)&=10 \quad \dots \quad \dots \quad (1) \\ x^2y+xy^2&=11 \quad \dots \quad \dots \quad (2) \end{aligned}\right\}$

[*Cal. F.A.* 1865 & *All. I.E.* 1892]

From (1), $xy+x+y=9$

From (2), $xy(x+y)=18$

Putting $x+y=u$ and $xy=v$, we have

$$\left.\begin{aligned} u+v&=9 \\ uv&=18 \end{aligned}\right\}$$

$\therefore$ $$(u+v)^2-4uv=81-72=9$$

$\therefore$ $$u-v=\pm 3.$$

Thus we have two pairs of equations, namely,

$$\left.\begin{aligned} u+v&=9 \\ u-v&=3 \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} u+v&=9 \\ u-v&=-3 \end{aligned}\right\}$$

Hence, we have $u=6$, $v=3$; or, $u=3$, $v=6$.

(i) Taking the first pair of values of u and v, namely, $u=6$ and $v=3$, we have $x+y=6$ and $xy=3$, whence we can obtain x and y. The two solutions are : $x=3\pm\sqrt{6}$, $y=3\mp\sqrt{6}$.

(ii) Taking the second pair of values of u and v, that is $u=3$ and $v=6$, we have $x+y=3$, and $xy=6$; which lead to two solutions : $x=\frac{1}{2}(3\pm\sqrt{-15})$, $y=\frac{1}{2}(3\mp\sqrt{-15})$.

Thus we have the four solutions :

$$\left.\begin{aligned} x&=3+\sqrt{6} \\ y&=3-\sqrt{6} \end{aligned}\right\} \quad \left.\begin{aligned} x&=3-\sqrt{6} \\ y&=3+\sqrt{6} \end{aligned}\right\} \quad \left.\begin{aligned} x&=\tfrac{1}{2}(3+\sqrt{-15}) \\ y&=\tfrac{1}{2}(3-\sqrt{-15}) \end{aligned}\right\} \quad \left.\begin{aligned} x&=\tfrac{1}{2}(3-\sqrt{-15}) \\ y&=\tfrac{1}{2}(3+\sqrt{-15}) \end{aligned}\right\}$$

Example 2. *Solve* : $$\left.\begin{aligned} x^2+y(x+1)&=17 \quad \cdots \quad \cdots \quad (1) \\ y^2+x(y+1)&=13 \quad \cdots \quad \cdots \quad (2) \end{aligned}\right\}$$

Adding (1) and (2) we have

$$x^2+y^2+2xy+x+y=30$$

or, $$(x+y)^2+(x+y)-30=0$$

or, $$\{(x+y)-5\}\{(x+y)+6\}=0$$

$\therefore$ $$x+y=5 \text{ or } -6 \quad \cdots \quad \cdots \quad (3)$$

Subtracting (2) from (1), we have

$$x^2-y^2+y-x=4$$

or, $$(x-y)\{(x+y)-1\}=4 \quad \cdots \quad \cdots \quad (4)$$

Hence from (3) and (4),

(i) when $x+y=5$, we have from (4)

$$(x-y)(5-1)=4, \text{ or } x-y=1.$$

$\therefore$ $$x=3 \text{ and } y=2.$$

(ii) when $x+y=-6$, we have from (4)

$$(x-y)(-6-1)=4, \text{ or } x-y=-\tfrac{4}{7}.$$

$\therefore$ $$x=-3\tfrac{2}{7}, \text{ and } y=-2\tfrac{5}{7}.$$

Thus the solutions are $x=3$, $y=2$; and $x=-3\frac{2}{7}$, $y=-2\frac{5}{7}$.

Example 3. *Solve* :

$$\left.\begin{aligned} 18y^x - y^{2x} &= 81 \quad \cdots \quad \cdots \quad (1)\\ 3^x &= y^2 \quad \cdots \quad \cdots \quad (2) \end{aligned}\right\}$$

From (1), $\quad y^{2x} - 18y^x + 81 = 0$

or, $\quad (y^x - 9)^2 = 0$

$\therefore \quad y^x = 9 \quad \cdots \quad \cdots \quad (3)$

or, $\quad (y^2)^{\frac{x}{2}} = 9$

Hence from (2), $\quad (3^x)^{\frac{x}{2}} = 9$

or, $\quad 3^{\frac{x^2}{2}} = 3^2$

$\therefore \quad \dfrac{x^2}{2} = 2$, or, $x^2 = 4 \quad \therefore x = \pm 2$.

Substituting in (3), when $x = 2$, $\quad y^2 = 9 \quad \therefore y = \pm 3$.

and when $x = -2$, $y^{-2} = \dfrac{1}{y^2} = 9$. $\therefore y = \pm \frac{1}{3}$.

Thus the solutions are : $x = 2$, $y = \pm 3$; and $x = -2$, $y = \pm \frac{1}{3}$.

EXERCISE 32.

Solve the following equations :

1. $x + y + xy = 27$, $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{2}$. [Cal. Int. 1939]
2. $x(x+1) + y(y+1) = 50$, $xy = 20$.
3. $x^2 + y(x+1) = 137$, $y^2 + x(y+1) = 205$. [Cal. F. A. 1894]
4. $\dfrac{x+y}{1-xy} = 3$, $\dfrac{x-y}{1+xy} = \dfrac{1}{3}$ [Cal. F. A. 1905]
5. $xy - \dfrac{x}{y} = a$, $xy - \dfrac{y}{x} = \dfrac{1}{a}$. [Cal. F. A. 1904]
6. $x^4 + y^4 = 14x^2y^2$, $x + y = a$. 7. $x^2 = 2x + 3y$, $y^2 = 3x + 2y$.
8. $4(x+y) = 3xy$, $x + y + x^2 + y^2 = 26$. [Mad. F. A. 1887]
9. $xy(x-a)(y-b) + ab(a-b)^2 = 0$, $x^2 + y^2 - (ax + by) = (a-b)^2$.
10. $(x^2 + y^2)(x+y) = 272$, $(x^2 - y^2)(x-y) = 32$. [All. I.E. 1894]
11. $x^2 + a^2 = y^2 + b^2 = (x+y)^2 + (a-b)^2$.
12. $6x + 5y = \dfrac{6}{x} + \dfrac{5}{y} + 29\frac{1}{3}$, $3x + 4y = \dfrac{3}{x} + \dfrac{4}{y} + 18\frac{2}{3}$.

[Mad. F. A.

13. $\dfrac{x^2+y^2}{xy}+x^2+y^2=13\frac{1}{3},\ \dfrac{xy}{x^2+y^2}+xy=3\frac{3}{10}$.
[Cal. F. A. 1903]

14. $x^4+x^2y^2+y^4=931,\ x^2+xy+y^2=49.$

15. $(ax+by)^2+(ay-bx)^2=2\left(\dfrac{a}{b}+\dfrac{b}{a}\right),\ \dfrac{x}{y}+\dfrac{y}{x}=2\left(\dfrac{a^2+b^2}{a^2-b^2}\right)$.

16. $bx+ay=a^2+b^2,\ \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=\dfrac{b^2}{a^2}+\dfrac{a^2}{b^2}$ [Pun. I. E. 1889]

17. $x^2+xy+y^2=13(x+y),\ x^2-xy+y^2=14(x-y).$

18. $x-\sqrt{x}=10-y,\ y-\sqrt{y}=11-x.$

19. $(x+1)\dfrac{x}{y}+(y+1)\dfrac{y}{x}=7,\ (x+y)\left(\dfrac{1}{x}+\dfrac{1}{y}\right)=4\frac{1}{2}.$

20. $\dfrac{x}{y+1}+\dfrac{y}{x+1}=\frac{2}{3},\ x^2+y^2=2.$ [Mad. F. A. 1891]

21. $\dfrac{bx}{y+b}+\dfrac{ay}{x+a}=\frac{1}{2}(a+b),\ \dfrac{x}{a}+\dfrac{y}{b}=2.$

22. $y^x=x^y,\ x^a=y^b.$ 23. $x^{x+y}=y^{4m},\ y^{x+y}=x^m.$

24. $x^y=y^2,\ y^{2y}=x^4.$ [Cal. Int. 1941]

25. $8.2^{xy}=4^y,\ 9^x.3^{xy}=\frac{1}{27}.$ [Cal. Int. 1942]

26. $x+y+3\sqrt{x+y}=x^2+y^2=10.$ [Pun. I. E. 1887]

27. $(ax+by)\left(\dfrac{x}{a}+\dfrac{y}{b}\right)=x^2+y^2+a^2+b^2,\ \dfrac{x}{a}+\dfrac{y}{b}=\sqrt{5}.$

28. $x+y+x^2+y^2=18,\ xy=6.$ [Cal. F. A. 1876]

29. $x^{\frac{1}{2}}+y^{\frac{1}{2}}=4,\ x^{\frac{3}{2}}+y^{\frac{3}{2}}=28.$ [Cal. F. A. 1868]

30. $\dfrac{x^2}{y^2}+\dfrac{x}{y}+\dfrac{y}{x}=\dfrac{27}{4}-\dfrac{y^2}{x^2},\ x-y=2.$ [Cal. F. A. 1865]

31. $x(9-xy)=y(xy-36),\ xy(3x+12y-xy)=108(x+y-3).$
[Mad. F. A. 1882]

32. $y+\sqrt{(x^2-1)}=2,\ \sqrt{(x+1)}+\sqrt{(x-1)}=\dfrac{2}{\sqrt{y}}$.
[Mad. F. A. 1889]

33. $x\sqrt{(1-y^2)}-y\sqrt{(1-x^2)}=xy-\sqrt{(1-x^2)}\sqrt{(1-y^2)}=\frac{1}{2}.$

34. $x^4+2x^3y+x^2y^2+2xy^3+y^4=41,\ \dfrac{x}{y}+\dfrac{y}{x}=\dfrac{5}{2}.$
[Cal. F. A. 1889]

35. $\dfrac{x^3+y^3}{x-y}-\dfrac{x^3-y^3}{x+y}=\dfrac{x^2+y^2}{\dfrac{1}{y}-\dfrac{1}{x}},\ \dfrac{\dfrac{x}{y}+\dfrac{y}{x}+1}{\dfrac{x}{y}+\dfrac{y}{x}-1}=\dfrac{2(x^3-y^3)}{x^3+y^3}.$

[Cal. F. A. 1875]

II. Equations involving Three Unknowns.

99. No general method can be laid down for the solution of simultaneous quadratic equations involving three (or more) unknowns ; nor in fact can such systems of equations always be solved. The process of solution, when solution is at all possible, consists essentially in *eliminating* two of the variables from the three equations, and thus getting an equation involving the third variable alone. If this last equation is soluble, then the third variable can be found, and thence the corresponding values of the other two variables.

We propose first to deal with systems of equations in which this elimination can be easily performed, and shall then solve certain typical systems of equations which are important in practice, and to which many other systems of equations can be reduced.

100. Two Equations of the First Degree : Method of Substitution.

When two out of the three simultaneous equations involving x, y and z are of the first degree and the third a quadratic, we may easily express x and y in terms of z by the help of these two equations, and then *substitute* these values in the third equation, which is thus reduced to a quadratic equation in z alone. Solve this quadratic in z, and thence find the corresponding values of x and y ; and the complete solution is obtained.

Example 1. *Solve* :

$$\left.\begin{aligned} x+2y+3z&=14 \quad \ldots \quad (1)\\ 2x+3y+z&=11 \quad \ldots \quad (2)\\ 3x^2+y^2+2z^2&=25 \quad \ldots \quad (3)\end{aligned}\right\}$$

Multiplying (1) by 3, and (2) by 2, and subtracting, we have

$$x-7z=-20 \qquad \therefore\ x=7z-20 \quad \cdots \quad (4)$$

Again, multiplying (1) by 2, and subtracting (2), we have

$$y+5z=17 \qquad \therefore\ y=17-5z \quad \ldots \quad (5)$$

Substituting from (4) and (5) in (3), we have

$$3(7z-20)^2+(17-5z)^2+2z^2=25$$

or, $$174z^2-1010z+1464=0$$

or, $$87z^2-505z+732=0 \qquad \text{[dividing by 2]}.$$

or, $$(z-3)(87z-244)=0$$

$$\therefore \qquad z=3 \text{ or } \tfrac{244}{87}.$$

Hence from (4), $x=1$ or $-\frac{32}{87}$

and from (5), $y=2$ or $\frac{259}{87}$.

Thus the solutions are :

$$x=1,\ y=2,\ z=3, \text{ and } x=-\tfrac{32}{87},\ y=\tfrac{259}{87},\ z=\tfrac{244}{87}.$$

Example 2. *Solve* : $\left.\begin{array}{ll} y+z : z+x : x+y=a : b : c & \cdots\ (1) \\ (y+z)^2+(z+x)^2+(x+y)^2=1 & \cdots\ (2) \end{array}\right\}$

From (1), we have [*Cal. F. A.* 1885.]

$$\frac{y+z}{a}=\frac{z+x}{b}=\frac{x+y}{c}=k \text{ (say)}$$

$$\therefore \quad y+z=ak,\ z+x=bk,\ x+y=ck \quad \cdots\ (3)$$

Substituting these values in (2), we have

$$a^2k^2+b^2k^2+c^2k^2=1 \quad \text{or} \quad k^2(a^2+b^2+c^2)=1$$

or, $$k^2=\frac{1}{a^2+b^2+c^2} \quad \therefore \quad k=\pm\frac{1}{\sqrt{a^2+b^2+c^2}} \quad \cdots(4)$$

Again, from (3), we have

$$(z+x)+(x+y)-(y+z)=bk+ck-ak$$

or, $$2x=(b+c-a)k$$

$\therefore$ from (4), $$x=\pm\frac{b+c-a}{2\sqrt{a^2+b^2+c^2}}.$$

Similarly, $$y=\pm\frac{c+a-b}{2\sqrt{a^2+b^2+c^2}}$$

and $$z=\pm\frac{a+b-c}{2\sqrt{a^2+b^2+c^2}}.$$

It is to be remarked that the + signs go together and the − signs go together, so that there are two sets of solutions.

NOTE. This system of equations is really equivalent to four equations, *viz.*, equation (2) and the three equations in (3) involving the four unknown quantities x, y, z and k; and the number of solutions is $1\times1\times1\times2$ or 2.

101. Two Equations of the First Degree : Method of Cross-multiplication.

If the two equations of the first degree have their constant term zero, the elimination can be readily effected by *cross-multiplication*. The following examples illustrate the method.

Example 1. *Solve* : $\left.\begin{aligned} 3x+\ y-5z&=0 \quad (1)\\ 7x-3y-9z&=0 \quad (2)\\ x^2+2y^2+3z^2&=23 \quad (3)\end{aligned}\right\}$ [*Cal. F. A.* 1873]

From (1) and (2), by cross-multiplication, we have

$$\frac{x}{1.(-9)-(-5)(-3)}=\frac{y}{(-5).7-3.(-9)}=\frac{z}{3(-3)-7.1}$$

or, $$\frac{x}{-24}=\frac{y}{-8}=\frac{z}{-16}$$

or, $$\frac{x}{3}=\frac{y}{1}=\frac{z}{2}=k \text{ (say)}.$$

Then $$x=3k,\ y=k,\ z=2k \quad \cdots \quad (4)$$

Substituting from (4) in (3), we have

$9k^2+2k^2+12k^2=23$, or, $23k^2=23$

or, $k^2=1$ $\therefore$ $k=\pm 1$.

Hence, $x=\pm 3,\ y=\pm 1,\ z=\pm 2$.

The two solutions, therefore, are :

$x=3,\ y=1,\ z=2$; and $x=-3,\ y=-1,\ z=-2$.

102. Equations involving x^2, y^2, z^2 only.

If the equations be of the form

$$\left.\begin{aligned} a_1x^2+b_1y^2+c_1z^2&=d_1\\ a_2x^2+b_2y^2+c_2z^2&=d_2\\ a_3x^2+b_3y^2+c_3z^2&=d_3\end{aligned}\right\}$$

they can be solved like linear simultaneous equations In fact, if we put $x^2=u$, $y^2=v$, $z^2=w$, the equations become linear in u, v, w.

Example 1. *Solve* : $3x^2-2y^2+5z^2=0 \quad \cdots \quad (1)$

$7x^2-3y^2-15z^2=0 \quad \cdots \quad (2)$

$5x^2+6y^2-\ 9z^2=132 \quad \cdots \quad (3)$

From (1) and (2) by cross-multiplication, we have

$$\frac{x^2}{(-2)(-15)-(-3).5}=\frac{y^2}{5.7-3(-15)}=\frac{z^2}{3(-3)-7(-2)}$$

or, $$\frac{x^2}{45}=\frac{y^2}{80}=\frac{z^2}{5} \quad \text{or,} \quad \frac{x^2}{9}=\frac{y^2}{16}=\frac{z^2}{1}=k^2 \text{ (say)}.$$

$\therefore$ $$x^2=9k^2,\ y^2=16k^2,\ z^2=k^2$$

or $$x=\pm 3k,\ y=\pm 4k,\ z=\pm k \quad \cdots \quad (4)$$

Substituting from (4) in (3), we have

$$45k^2+96k^2-9k^2=132,$$

or, $$132k^2=132,$$

or, $$k^2=1 \quad \therefore\ k=\pm 1$$

Hence the 3 solutions are : $x=\pm 3$, $y=\pm 4$, $z=\pm 1$.

EXERCISE 33.

Solve the following equations :

1. $2x-7y+6z=9,\ 4x+3y-2z=19,\ 4x^2+3y^2-2x^2=67.$
2. $x+4y+3z=17,\ 3x+3y+z=16,\ 2x^2+y^2+3z^2=20.$
3. $\dfrac{x}{a}+\dfrac{y}{b}=\dfrac{x}{a}+\dfrac{z}{c}=1,\ yz=bc.$
4. $x+3y-2z=0,\ 7x-5y+z=0,\ 3x^2+5y^2+z^2=73.$
5. $3x+y-2z=0,\ 4x-y-3z=0,\ 3x^2+5y^2-z^2=124.$
6. $3x+4y-6z=0,\ 6x-7y+2z=0,\ x^3+y^3+z^3=216.$
7. $x-2y+z=0,\ 9x-8y+3z=0,\ yz+zx+xy=23.$
8. $x^2+2y^2+3z^2=36,\ 5x^2-2y^2+z^2=6,\ x+y-z=0.$
9. $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}=\dfrac{1}{ax+by+cz}.$
10. $ax+by+cz=0,\ \dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=0,$

$$\frac{x^2-a^2}{b^2-c^2}+\frac{y^2-b^2}{c^2-a^2}+\frac{z^2-c^2}{a^2-b^2}=0.$$

103. Some Important Types of Equations.

We proceed now to solve some important types of simultaneous equations to which many systems of equations can be reduced.

(i) Let us take the system

$$\left.\begin{aligned} yz&=a^2 \quad \cdots \quad (1)\\ zx&=b^2 \quad \cdots \quad (2)\\ xy&=c^2 \quad \cdots \quad (3)\end{aligned}\right\}$$

Multiplying (1), (2), and (3) together, we have

$$x^2y^2z^2=a^2b^2c^2$$

$$\therefore \quad xyz=\pm abc \quad \cdots \quad (4)$$

Dividing (4) by (1), $x = \pm \frac{bc}{a}$

Dividing (4) by (2), $y = \pm \frac{ca}{b}$

Dividing (4) by (3), $z = \pm \frac{ab}{c}$

In the double signs above, the + signs go together and the − signs together.

Otherwise : Multiplying (2) and (3) and dividing by (1),

we have $x^2 = \frac{b^2c^2}{a^2}$ $\therefore$ $x = \pm \frac{bc}{a}$

Similarly, $y = \pm \frac{ca}{b}$, $z = \pm \frac{ab}{c}$.

(ii) Let us solve the system

$$\left.\begin{aligned} x+y+z &= a & \cdots \quad \cdots \quad (1)\\ yz+zx+xy &= b^2 & \cdots \quad \cdots \quad (2)\\ xyz &= c^3 & \cdots \quad \cdots \quad (3) \end{aligned}\right\}$$

We know that if α, β, γ be three roots of a cubic equation $x^3+px^2+qx+r=0$, then

$$\left.\begin{aligned} \alpha+\beta+\gamma &= -p\\ \beta\gamma+\gamma\alpha+\alpha\beta &= q\\ \alpha\beta\gamma &= -r \end{aligned}\right\} \quad (4) \text{ [See Art. 71, Note]}$$

We see therefore that the values of x, y, z which satisfy the given system of equations must be the roots of the cubic equation $X^3 - aX^2 + b^2X - c^3 = 0$ ... ... (5)

The process therefore consists in solving the cubic equation (5) and the roots of this cubic equation will constitute the solution of the given system. It should be remarked however that any one of the variables x, y, z may stand for any of these three roots, so that the number of sets of solutions is 6, as expected.

Otherwise : The solution may be effected also by *elimination* as follows :

From (1), $y+z = a-x$

From (3), $yz = \frac{c^3}{x}$

Substituting these in (2) we have

$$\frac{c^3}{x}+x(a-x)=b^2$$

or, $$x^3-ax^2+b^2x-c^3=0$$

which is the same cubic equation as (5) above. Solving this, we get three values of x and thence the corresponding values of y and z.

Example 1. *Solve* : $\left.\begin{aligned} x(y+z)=a \\ y(z+x)=b \\ z(x+y)=c \end{aligned}\right\}$

These equations may be written thus :

$$xy+zx=a \quad \cdots (1)$$
$$yz+xy=b \quad \cdots (2)$$
$$zx+yz=c \quad \cdots (3)$$

Adding and dividing by 2, we have

$$xy+yz+zx=\tfrac{1}{2}(a+b+c) \quad \cdots (4)$$

Subtracting (1), (2) and (3) respectively from (4), we have

$$yz=\tfrac{1}{2}(b+c-a) \quad \cdots (5)$$
$$zx=\tfrac{1}{2}(c+a-b) \quad \cdots (6)$$
$$xy=\tfrac{1}{2}(a+b-c) \quad \cdots (7)$$

Now, multiplying together (5), (6) and (7), and extracting the square root, we have

$$xyz=\pm\frac{1}{2\sqrt{2}}\sqrt{(b+c-a)(c+a-b)(a+b-c)} \quad \cdots (8)$$

Hence, dividing (8) by (5), (6) and (7) respectively, we have

$$x=\pm\sqrt{\frac{(a+b-c)(c+a-b)}{2(b+c-a)}}$$

$$y=\pm\sqrt{\frac{(a+b-c)(b+c-a)}{2(c+a-b)}}$$

$$z=\pm\sqrt{\frac{(b+c-a)(c+a-b)}{2(a+b-c)}}$$

Example 2. *Solve* :
$$\left.\begin{aligned}(x+y)(x+z)&=bc \quad \ldots \quad \ldots \quad (1)\\(y+z)(y+x)&=ca \quad \ldots \quad \ldots \quad (2)\\(z+x)(z+y)&=ab \quad \ldots \quad \ldots \quad (3)\end{aligned}\right\}$$

Multiplying these equations together, we have
$$(x+y)^2(y+z)^2(z+x)^2=a^2b^2c^2$$
or,
$$(x+y)(y+z)(z+x)=\pm abc \quad \ldots \quad (4)$$

Dividing (4) by (1), (2), and (3) respectively, we have
$$\left.\begin{aligned}y+z&=\pm a\\z+x&=\pm b\\x+y&=\pm c\end{aligned}\right\} \quad \ldots \quad \ldots \quad (5)$$

Adding the equations (5) and dividing by 2,
$$x+y+z=\pm\tfrac{1}{2}(a+b+c) \ldots \quad \ldots \quad (6)$$

Hence, subtracting the equations (5) separately from (6), we have $x=\pm\{\frac{1}{2}(a+b+c)-a\}=\pm\frac{1}{2}(b+c-a)$;
$$y=\pm\{\tfrac{1}{2}(a+b+c)-b\}=\pm\tfrac{1}{2}(c+a-b);$$
$$z=\pm\{\tfrac{1}{2}(a+b+c)-c\}=\pm\tfrac{1}{2}(a+b-c).$$

Example 3. *Solve* :
$$\left.\begin{aligned}yz+cy+bz&=l^2 \quad \ldots \quad \ldots \quad (1)\\zx+az+cx&=m^2 \quad \ldots \quad \ldots \quad (2)\\xy+bx+ay&=n^2 \quad \ldots \quad \ldots \quad (3)\end{aligned}\right\}$$

From (1), by adding bc to both sides, we have
$$(y+b)(z+c)=l^2+bc \quad \ldots \quad \ldots \quad (4)$$
Similarly from (2), $(z+c)(x+a)=m^2+ca \quad \ldots \quad \ldots \quad (5)$

and from (3), $(x+a)(y+b)=n^2+ab \quad \ldots \quad \ldots \quad (6)$

Multiplying (4), (5), and (6) together, we have
$$(x+a)^2(y+b)^2(z+c)^2=(l^2+bc)(m^2+ca)(n^2+ab)$$
or, $(x+a)(y+b)(z+c)=\pm\sqrt{(l^2+bc)(m^2+ca)(n^2+ab)}$ (7)

Hence from (4) and (7), we have, by division,
$$x+a=\pm\sqrt{\frac{(m^2+ca)(n^2+ab)}{l^2+bc}}$$
or,
$$x=-a\pm\sqrt{\frac{(m^2+ca)(n^2+ab)}{l^2+bc}}$$
Similarly,
$$y=-b\pm\sqrt{\frac{(l^2+bc)(n^2+ab)}{m^2+ca}}$$
and
$$z=-c\pm\sqrt{\frac{(l^2+bc)(m^2+ca)}{n^2+ab}}.$$

Example 4. *Solve :*
$$\left.\begin{array}{ll} x(x+y+z)=a^2 & (1) \\ y(x+y+z)=b^2 & (2) \\ z(x+y+z)=c^2 & (3) \end{array}\right\}$$
[*Cal. F. A.* 1907]

Adding the given equations, we have

$$(x+y+z)^2=a^2+b^2+c^2$$

$$\therefore \quad x+y+z=\pm\sqrt{a^2+b^2+c^2} \quad \ldots \quad \ldots \quad (4)$$

Hence dividing the equations (1), (2) and (3) successively by (4), we have

$$x=\pm\frac{a^2}{\sqrt{a^2+b^2+c^2}}$$

$$y=\pm\frac{b^2}{\sqrt{a^2+b^2+c^2}}$$

$$z=\pm\frac{c^2}{\sqrt{a^2+b^2+c^2}}$$

Example 5. *Solve :*
$$\left.\begin{array}{ll} yz=a(y+z) & (1) \\ zx=b(z+x) & (2) \\ xy=c(x+y) & (3) \end{array}\right\}$$
[*Cal. F. A.* 1874]

Dividing (1) by ayz, we have

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{a} \quad \ldots \quad \ldots \quad (4)$$

Similarly from (2) and (3), we have

$$\frac{1}{z}+\frac{1}{x}=\frac{1}{b} \quad \ldots \quad \ldots \quad (5)$$

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{c} \quad \ldots \quad \ldots \quad (6)$$

Adding together, $2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

$$\therefore \quad \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \ldots (7)$$

Subtracting (4) from (7), $\frac{1}{x}=\frac{1}{2}\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)$

Subtracting (5) from (7), $\frac{1}{y}=\frac{1}{2}\left(\frac{1}{c}+\frac{1}{a}-\frac{1}{b}\right)$

Subtracting (6) from (7), $\frac{1}{z}=\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)$

Hence, $x=\frac{2abc}{ca+ab-bc}$, $y=\frac{2abc}{ab+bc-ca}$, $z=\frac{2abc}{bc+ca-ab}$.

Example 6. *Solve* : $\left.\begin{array}{ll} x^2-yz=a & (1) \\ y^2-zx=b & (2) \\ z^2-xy=c & (3) \end{array}\right\}$ [*All. I. E.* 1929]

From the given equations, we have

$$a^2-bc=(x^2-yz)^2-(y^2-zx)(z^2-xy)=x(x^3+y^3+z^3-3xyz)$$
$$b^2-ca=(y^2-zx)^2-(z^2-xy)(x^2-yz)=y(x^3+y^3+z^3-3xyz)$$
$$c^2-ab=(z^2-xy)^2-(x^2-yz)(y^2-zx)=z(x^3+y^3+z^3-3xyz)$$

$$\therefore \qquad \frac{x}{a^2-bc}=\frac{y}{b^2-ca}=\frac{z}{c^2-ab}$$

because each of them $=\dfrac{1}{x^3+y^3+z^3-3xyz}$.

Let each of these ratios $=k$; then

$$x=k(a^2-bc)\ ;\ y=k(b^2-ca)\ ,\ z=k(c^2-ab) \quad \cdots \quad (4)$$

Substituting these values of x, y, z in (1) we have

$$k^2\{(a^2-bc)^2-(b^2-ca)(c^2-ab)\}=a$$

or, $$k^2a(a^3+b^3+c^3-3abc)=a$$

$$\therefore \qquad k=\pm\frac{1}{\sqrt{a^3+b^3+c^3-3abc}}.$$

Hence from the relations (4), we have

$$x=k(a^2-bc)=\pm\frac{a^2-bc}{\sqrt{a^3+b^3+c^3-3abc}}$$

and two similar expressions for y and z.

EXERCISE 34.

Solve the following equations :

1. $yz=a^2,\ zx=b^2,\ xy=c^2$. 2. $xyz=\dfrac{y+z}{a}=\dfrac{z+x}{b}=\dfrac{x+y}{c}$.

3. $x^2yz=a,\ y^2zx=b,\ z^2xy=c$.
4. $(x+y)(x+z)=56,\ (y+z)(y+x)=77,\ (z+x)(z+y)=88$.
5. $x(y+z)=5,\ y(z+x)=8,\ z(x+y)=9$. [Cal. Int. 1938]
6. $xy+zx=a(b+c),\ yz+xy=b(c+a),\ zx+yz=c(a+b)$.
7. $x(x+y+z)=97,\ y(x+y+z)=121,\ z(x+y+z)=143$.
8. $x(y+z-x)=a^2,\ y(z+x-y)=b^2,\ z(x+y-z)=c^2$.
9. $xy+x+y=23,\ yz+y+z=27,\ zx+z+x=41$.
[Cal. Int. 1912, 1945 & All. I. E. 1927]

10. $xyz=\dfrac{x+y}{3}=\dfrac{y+z}{4}=\dfrac{z+x}{5}$. [All. I. E. 1892]

11. $xz+y=7z,\ yz+x=8z,\ x+y+z=12.$ [Cal. Int. 1939]

12. $yz+m(y+z)=a,\ zx+m(z+x)=b,\ xy+m(x+y)=c.$

13. $\dfrac{xy}{x+y}=1,\ \dfrac{yz}{y+z}=2,\ \dfrac{zx}{z+x}=3.$ [Cal. F. A. 1876]

14. $ax+yz=ay+zx=az+xy=p^2.$

15. $x(a+y+z)=y(a+z+x)=z(a+x+y)=3a(x+y+z).$

16. $a^2-x^2=b^2-y^2=c^2-z^2=yz+zx+xy.$

17. $(y+z)^2=a^2+x^2,\ (z+x)^2=b^2+y^2,\ (x+y)^2=c^2+z^2.$

18. $x^2-(y-z)^2=a^2,\ y^2-(z-x)^2=b^2,\ z^2-(x-y)^2=c^2.$

19. $x^2-y^2-z^2+2yz=3,\ -x^2+y^2-z^2+2xz=-9,$
$-x^2-y^2+z^2+2xy=-3.$ [Cal. F. A. 1866]

20. $x^2-yz=16,\ y^2-zx=-14,\ z^2-xy=1.$

21. $ax+by+cz=0,\ \dfrac{1}{yz}+\dfrac{1}{zx}+\dfrac{1}{xy}+1=0,$
$a^3x+b^3y+c^3z+abcxyz=0.$ [Bom. P. E. 1896]

22. $x+\dfrac{1}{y}=\dfrac{3}{2},\ \ y+\dfrac{1}{z}=\dfrac{7}{3},\ \ z+\dfrac{1}{x}=4.$ [Cal. Int. 1942]

23. $\dfrac{yz}{bz+cy}=\dfrac{zx}{cx+az}=\dfrac{xy}{ay+bx}=\dfrac{x^2+y^2+z^2}{a^2+b^2+c^2}.$

24. $x^2+y^2+z^2=84,\ x+y+z=14,\ y^2=zx.$ [Cal. Int. 1940]

25. $x+y+z=1\frac{1}{12},\ yz+zx+xy=\frac{9}{24},\ xyz=\frac{1}{24}.$ [All. I. E. 1925]

26. $x+y+z=9,\ yz+zx+xy=26,\ x^3+y^3+z^3=99.$

27. $x+y+z=15,\ xyz=105,\ x^3+y^3+z^3=495.$ [All. I.E. 1930]

28. $x^3+y^3+z^3=a^3,\ x^2+y^2+z^2=a^2,\ x+y+z=a.$
[All. I. E. 1928]

29. $x+y+z=3,\ yz+zx+xy=-1,\ \dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=\dfrac{1}{3}.$
[Cal. F. A. 1879]

30. $x+y+z=1,\ x^2+y^2+z^2+6xy=0,\ \dfrac{x}{y+z}+\dfrac{y}{z+x}+\dfrac{z}{x+y}=0.$
[Cal. F. A. 1862]

31. $y^2+yz+z^2=7,\ z^2+zx+x^2=13,\ x^2+xy+y^2=9.$
[All. I. E. 1921]

32. $x^2+y^2+z^2=14,\ x^3+y^3+z^3=36,\ xyz=6.$

33. $x^3+y^3+z^3=3xyz,\ 3a-x+z=3b-y+x=3c-z+y.$

34. $x+y+z=10,\ yz+zx+xy=33,\ (y+z)(z+x)(x+y)=294.$

35. $(x+y+z)(yz+zx+xy)=69,\ (y+z)(z+x)(x+y)=60,$
$x^3+y^3+z^3=36.$

36.
$$\left.\begin{aligned} yz+zx+xy&=26,\\ yz(y+z)+zx(z+x)+xy(x+y)&=162,\\ yz(y^2+z^2)+zx(z^2+x^2)+xy(x^2+y^2)&=588.\end{aligned}\right\}$$

CHAPTER X

GRAPHS OF QUADRATIC EQUATIONS.

104. General Equation of the Second Degree.

The general equation of the second degree involving x and y may be written in the form

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

In order to draw the graph of such an equation, we have first to see if the quadratic expression involved is or is not resoluble into linear factors; if it is, then the graph consists of *a pair of straight lines*; while if it is not, the graph becomes either a *circle* or one or other of the curves known as the *conic sections*, as will be shown in the following Articles.

105. A Pair of Straight Lines.

When an equation of the second degree in x and y can be expressed in the form

$$(ax+by+c)(a'x+b'y+c')=0 \quad \dots \quad \dots \quad (1)$$

that is, in the form of a product of two expressions of the first degree in x and y equated to zero, then each of the equations

$$ax+by+c=0 \quad \dots \quad \dots \quad (2)$$

and

$$a'x+b'y+c'=0 \quad \dots \quad \dots \quad (3)$$

represents a straight line, and the graph of the equation (1) will be the pair of straight lines represented by the equations (2) and (3). For, the co-ordinates of any point on either of the straight lines (2) and (3) make either $ax+by+c=0$ or $a'x+b'y+c'=0$, and therefore they also satisfy the equation (1). Hence the graph of an expression of the second degree in x and y which can be resolved into two linear factors, equated to zero, is the *pair of straight lines* whose separate equations are the linear factors each equated to zero.

Example 1. *Draw the graph of* $x^2+2x-15=0$.

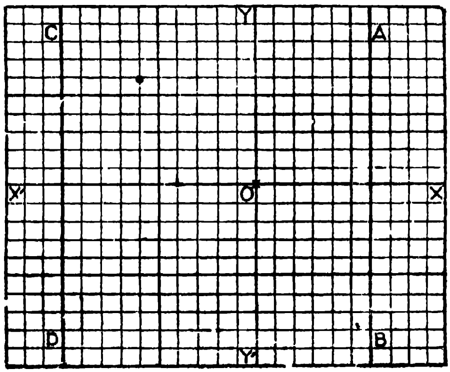

Fig. 1.

Since the given equation can be written as $(x-3)(x+5)=0$, the graph required is the pair of straight lines represented by the equations $x-3=0$ and $x+5=0$, *i.e.*, $x=3$ and $x=-5$.

Taking 2 small divisions for the unit along the x-axis, the required graph is the pair of lines, AB and CD [Fig. 1].

They are evidently parallel to the axis of y.

NOTE 1. Similarly, it is clear that an equation of the second degree involving y alone will be graphically represented by two straight lines parallel to the axis of x.

NOTE 2. From the above it is clear that the graph of an equation of the second degree involving only one unknown quantity represents a pair of straight lines, both parallel to the axis of y when the unknown quantity involved is x, and both parallel to the axis of x when the unknown quantity involved is y. If however the quadratic expression involving one unknown quantity only equated to zero cannot be resolved into two real factors, the graph is in that case *imaginary* and cannot be actually drawn, *e.g.*, $x^2+a^2=0$, or, $y^2+b^2=0$.

Examle 2. *Draw the graph of*

$$3x^2+5xy-2y^2-53x+13y+34=0.$$

The given equation can be resolved into two linear factors and is equivalent to $(x+2y-17)(3x-y-2)=0$.

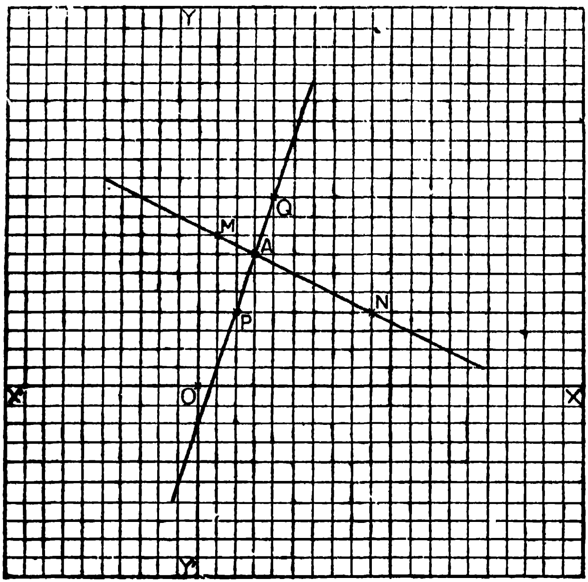

Fig. 2.

Hence, the graph of the equation is the pair of straight lines represented by the equations :

$$x+2y-17=0 \quad \ldots \quad \ldots \quad (1)$$

and $$3x-y-2=0 \quad \ldots \quad \ldots \quad (2)$$

Since the graph of the equation (1) passes through the points (1, 8), (9. 4), and the graph of the equation (2) passes through the points (2, 4), (4, 10), taking 1 small division for the unit of length along both the axes, the required graph is the pair of straight lines MN and PQ, as shown in Fig. 2. These two lines intersect at A, and the co-ordinates of the point of intersection are (3, 7).

NOTE. When the quadratic expression equated to zero is a *perfect square, i. e.,* of the form $(ax+by+c)^2=0$, the graph of the equation is a pair of coincident straight lines, which in drawing is equivalent to a single straight line represented by the equation $ax+by+c=0$. Thus, the graph of the equation $x^2+4xy+4y^2-2x-4y+1=0$ which is the same as $(x+2y-1)^2=0$, is a pair of coincident straight lines and is the same as the straight line represented by the equation $x+2y-1=0$.

EXERCISE 35.

Draw the graph of the following equations :

1. $x^2=0$.
2. $x^2-4=0$.
3. $y^2-9=0$.
4. $x^2-7x+12=0$.
5. $2x^2+5x-3=0$.
6. $3y^2+y-10=0$.
7. $x^2-6x+9=0$.
8. $14y^2+23y-30=0$.
9. $2x^2+5xy+3y^2=0$.
10. $4x^2+3xy-7y^2=0$.
11. $x^2+6xy+9y^2=0$.
12. $2x^2+7xy+6y^2+3x+5y+1=0$.
13. $x^2+xy-6y^2-x+17y-12=0$.
14. $4x^2-12xy+9y^2+20x-30y+25=0$.
15. Show that the graph of the equation $5x^2-4x+1=0$ is impossible.
16. Show that the two straight lines represented by the equation $x^2-y^2=0$ are at right angles to each other.

106. The Circle and the Conic Sections.

When an expression of the second degree in x and y cannot be resolved into two linear factors, the graph of such an expression equated to zero will be any one of the following curves, namely, a *circle*, a *parabola*, an *ellipse* or an *hyperbola*. The last three curves are called *conic sections*, or more shortly, *conics*, as they can be obtained by the section of a cone by a plane. (The circle, too, can be obtained by the section of a cone by a plane, though it is not usually called a conic.) We shall now proceed to examine the graphs of such equations.

(*a*) *Circle.*

Example 1. *Draw the graph of* $x^2+y^2=25$.

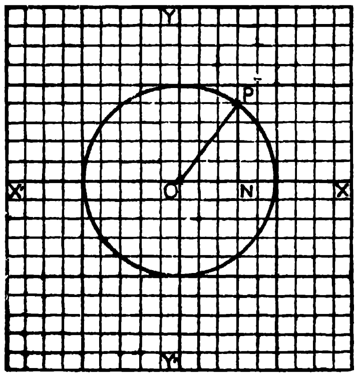

Fig. 3.

Let one small division represent the unit of length, and let P be any point on the graph so that its co-ordinates are ON and PN respectively [Fig. 3]. Then since P is a point on the graph, from the given equation, we have

$$ON^2+PN^2=25$$

but $$ON^2+PN^2=OP^2$$

$$OP^2=25$$

i.e., $$OP=5.$$

Thus, P being any point on the graph, the curve is such that the distance of any point on it from the origin is always equal to 5, a constant quantity. Hence the graph of the equation $x^2+y^2=25$ is a circle whose centre is the origin and radius equal to 5 units of length.

NOTE. As in the above example, exactly in the same way, the graph of any other equation of the form

$$x^2+y^2=a^2,$$

will be found to be a circle whose centre is the origin and radius equal to a.

Example 2. *Draw the graph of* $x^2 + y^2 = 29$.

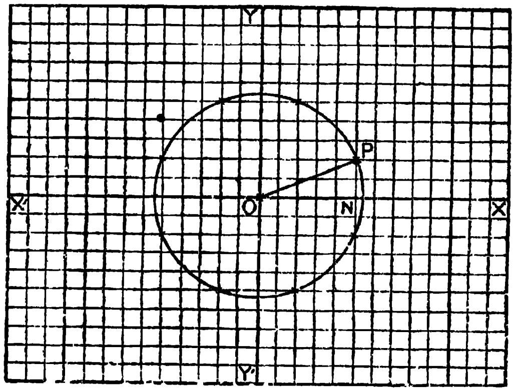

Fig. 4.

The graph of this equation is evidently a circle whose centre is the origin and radius equal to $\sqrt{29}$. Now, when $x=5$ and $y=2$, the equation is satisfied, and therefore $P(5, 2)$ is a point on the graph ; also $OP^2 = 5^2 + 2^2 = 29$; or, $OP = \sqrt{29}$. Hence the circle drawn with centre O and radius OP, is the graph required, as shown in Fig. 4.

Example 3. *Draw the graph of* $(x-2)^2 + (y-3)^2 = 16$.

Let two small divisions represent the unit of length. Let C be the point (2, 3), and P be any point on the graph whose co-ordinates are (α, β), suppose. Draw PN perpendicular to the line through C and parallel to the x-axis, so that $CN = \alpha - 2$ and $PN = \beta - 3$ [Fig. 5].

Since $P(\alpha, \beta)$ is any point on the graph, we have $(\alpha-2)^2 + (\beta-3)^2 = 16$, that is, $CN^2 + PN^2 = 16$; but CNP being a right-angled triangle, $CN^2 + PN^2 = CP^2$.

$$\therefore \qquad CP^2 = 16, \text{ or, } CP = 4.$$

Thus, P being any point on the graph, the curve is such that the distance of any point on it from C (2, 3) which is a fixed point, is equal to 4, a constant quantity. Hence the graph of the equation is a circle whose centre is the point (2, 3), and radius equal to 4 units of length.

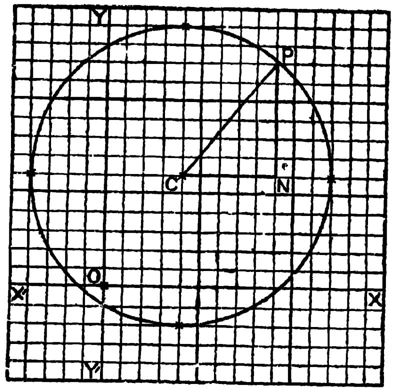

Fig. 5.

NOTE 1. As in the above example, exactly in the same way, the graph of any other equation which can be reduced to the form

$$(x-h)^2+(y-k)^2=a^2$$

will be found to be a circle whose centre is the point (h, k) and radius equal to a.

NOTE 2. If we have to draw the graph of an equation like $x^2+y^2-6x+10y-2=0$ which can be easily reduced to the form $(x-3)^2+(y+5)^2=36$, its graph will be a circle whose centre is the point $(3, -5)$ and radius equal to 6 units of length. More generally, the graph of any equation of the form

$$x^2+y^2+2ax+2by+c=0$$

which can be easily reduced to the form $(x+a)^2+(y+b)^2=a^2+b^2-c$, is a circle whose centre is $(-a, -b)$ and radius is $\sqrt{a^2+b^2-c}$. Hence, *the graphs of all equations in which the co-efficients of x^2 and y^2 are equal and there is no term involving xy, are circles.*

(b) *Parabola.*

Example 4. *Draw the graph of $x^2=4y$.* [*Cal.* 1926]

Writing the given equation in the form $y=\frac{1}{4}x^2$, we have the following series of values of x and y satisfying the equation, namely :

when	$x=0,$	$\pm 1,$	$\pm 2,$	$\pm 3,$	$\pm 4,$	± 5········
we have	$y=0,$	·25,	1,	2·25,	4,	6·25········

Taking two small divisions for the unit of length and plotting the points whose co-ordinates are given by the

above pairs of values of x and y, let us draw a curve through these points free-hand [Fig. 6]. This curve is the required graph of this equation.

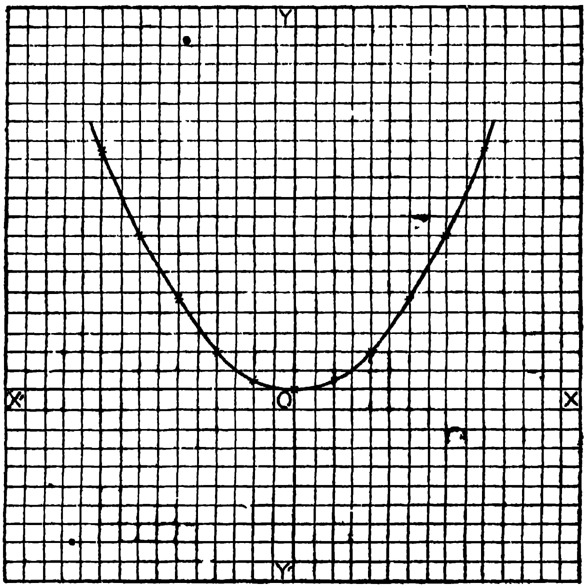

Fig. 6.

It is also evident that for any real value of x, $\frac{1}{4}x^2$ is always positive, and therefore y cannot be negative for any real value of x; hence there is no part of the curve lying below the axis of x.

As x increases either in the positive direction, or in the negative direction, y is always positive and increases with x, and since there is no limit to this increase of x and y, both the portions of the curve extend to infinity towards the positive direction of the y-axis.

A curve of this class is called a *parabola*.

NOTE 1. A curve is said to be *symmetrical* with regard to a straight line when all chords of the curve drawn perpendicular to the straight line are bisected by it and the straight line which bisects the chords is called the *axis of symmetry*. Thus, if the curve is folded about its axis of symmetry, the two branches of the curve lying on opposite sides of the axis will exactly coincide with each other. Since in the case of the above parabola, corresponding to any value of y, there are two values of x which are equal in magnitude but opposite in sign, it is clear that the curve is symmetrical about the axis of y. This line is called the *axis* of the parabola and the point O at which the axis meets the parabola is called the *vertex* of the parabola.

NOTE 2. If we want to draw the graph of the equation $4y = -x^2$, since for all real values of x, x^2 is positive and therefore y is always negative and cannot be positive for any real value of x, the graph of the equation will be a parabola passing through the origin and lying entirely below the axis of x; there is no part of the curve above the axis of x. The curve extends downwards to infinity and is symmetrical about the axis of y.

NOTE 3. As in the above example the graph of any equation of the form

$$x^2 = 4ay$$

will be found to be a parabola, whose vertex is at the origin, and whose axis is along the y-axis.

Example 5. *Draw the graph of $y^2 = x$.*

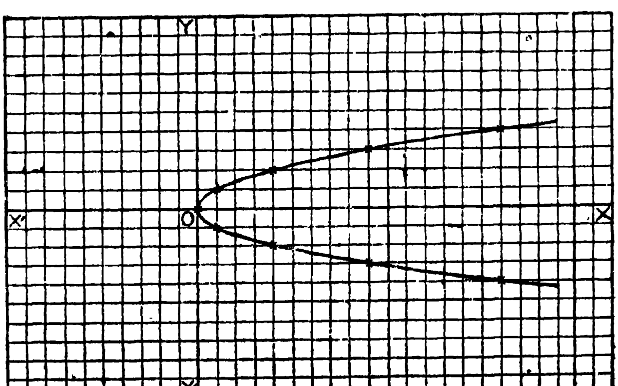

Fig. 7.

From the given equation,

when	$x=0$,	1,	4,	9,	16.........
we have	$y=0$,	± 1,	± 2,	± 3,	± 4.........

Let us take 1 small division = unity along OX, and 1 small division = unity along OY. Plotting the series of points and drawing a curve through them free-hand we get the graph required as in Fig. 7. This curve, too, is a *parabola.*

NOTE 1. Since $y=\pm\sqrt{x}$, that is for a given value of x we get two equal and opposite values of y the graph is symmetrical about the axis of x. Also y^2, being a square, is positive for real values of y, and hence y cannot be real for any negative value of x; hence there is no part of the curve on the left side of the axis of y. Thus, the parabola lies entirely on the right side of the axis of y. Its vertex is at the origin and its axis is the axis of x. Both the portions of the curve lying on opposite sides of the x-axis extend to infinity towards the positive direction of the x-axis.

NOTE 2. As in the above example, the graph of any equation of the form

$$y^2=4ax$$

will be found to be a parabola whose vertex is at the origin and whose axis is along the x-axis.

NOTE 3. *Square Roots graphically determined.*

The graph of $y^2=x$ may be used to determine graphically the square root of a number.

Suppose we want to determine the square root of 11. From the curve in Fig. 7, when $x=11$, the corresponding value of y is nearly 3·3, and hence the required square root is 3·3 nearly.

Example 6. *Draw the graph of* $y=2x^2-12x+13$.

From the given equation, it is easy to see that

when	$x=-1$,	0,	·5,	1,	2,	3,	4,	5,	5·5,	6,
we have	$y=27$,	13,	7·5,	3,	-3,	-5,	-3,	3,	7·5,	13.

Let us take 1 small division = unity along OX, and 1 small division = unity along OY, and plot the points given above. The graph required is the curve drawn freely through these points [Fig. 8]. This curve, too, is a *parabola.*

NOTE 1. From the graph drawn in Fig. 8. it is clear that $y=0$ when $x=1·4$ or 4·6 units. This shows that the roots of the quadratic equation $2x^2-12x+13=0$ are 1·4 and 4·6 nearly, that is, the abscissæ of the points where the parabola is cut by the axis of x.

NOTE 2. Writing the given equation in the form $y=2(x-3)^2-5$, it is clear that for all real values of x, $2(x-3)^2$ is always positive, and therefore y is always greater than -5 for any real value of x, except when $x=3$, in which case $y=-5$. This shows that y is minimum when $x=3$, and the minimum value of the expression $2x^2-12x+13$ is -5, as is also evident from the graph.

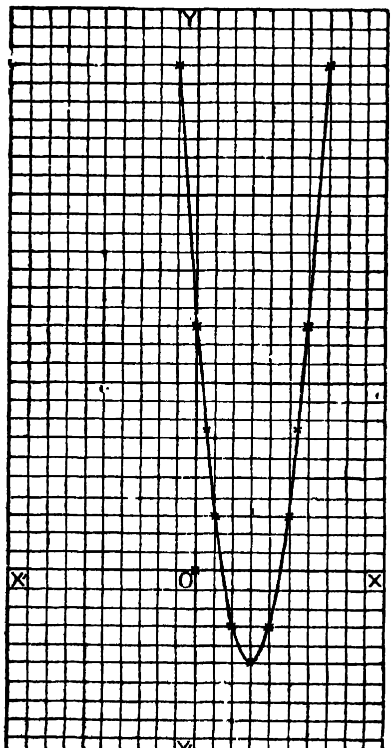

Fig.

NOTE 3. It is also clear from the figure that the vertex of the parabola is the point $(3, -5)$, and the curve is symmetrical about the line whose equation is $x=3$.

NOTE 4. From the last three examples it should be observed that the graph of any equation of either of the forms

$$y=ax^2+bx+c, \text{ or, } x=ay^2+by+c,$$

where a, b, c may have any values excepting $a=0$, is a parabola. The two roots of the quadratic equation $ax^2+bx+c=0$ are the abscissæ of the points at which the axis of x cuts the parabola $y=ax^2+bx+c$.

(c) *Ellipse.*

Example 7. *Draw the graph of* $9x^2+25y^2=225$.

Writing the given equation in the form

$$y=\pm\frac{3}{5}\sqrt{25-x^2}$$

we tabulate the values of x and y satisfying the equation in the following way. Thus,

when	$x=$	0,	± 2,	± 3,	± 4,	± 5,	…………
we have	$y=$	± 3,	$\pm 2{\cdot}75$,	$\pm 2{\cdot}4$,	$\pm 1{\cdot}8$,	0,	…………

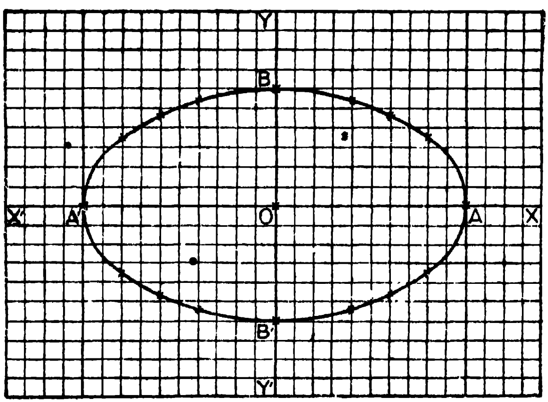

Fig. 9.

Thus the series of points $(0, 3)$, $(0, -3)$, $(2, 2{\cdot}75)$, $(2, -2{\cdot}75)$ $(-2, 2{\cdot}75)$, $(-2, -2{\cdot}75)$, etc., lie on the graph.

Taking 2 small divisions for the unit of length, and plotting these points, the required graph is the curve drawn

freely through these points as in Fig. 9. A curve of this class is called an *Ellipse.*

NOTE 1. From the equation $y=\pm\frac{3}{5}\sqrt{25-x^2}$, when x^2 is less than 25, *i.e.*, when x lies between -5 and $+5$, $25-x^2$ is positive and therefore y is real; for any value of x not lying between -5 and $+5$, $25-x^2$ is negative, and therefore y is imaginary. Hence the curve lies entirely between the lines $x=-5$ and $x=5$. Similarly, writing the same equation in the form $x=\pm\frac{5}{3}\sqrt{9-y^2}$, it can be easily shown that the curve lies entirely between the lines $y=-3$ and $y=+3$. Therefore an ellipse is a *closed curve.* Again, since $y=\pm\frac{3}{5}\sqrt{25-x^2}$, for a given value of x we get two equal and opposite values of y, hence the curve is symmetrical about the x-axis. Similarly since $x=\pm\frac{5}{3}\sqrt{9-y^2}$, for a given value of y there are two equal and opposite values of x, and hence the curve is also symmetrical about the y-axis. The axes AOA' and BOB' are respectively called the *major* and *minor axes*, and O the *centre*, and A and A' the *vertices* of the ellipse.

NOTE 2. Generally, the graph of an equation of the form $ax^2+by^2=c$, where a, b and c are all positive, is an ellipse. The equation of an ellipse is usually written in the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where a and b are the lengths of the semi-axes of the ellipse.

NOTE 3. In the equation $ax^2+by^2=c$, if $a=b$, the equation reduces to $x^2+y^2=\frac{c}{a}$, and its graph is a circle; hence *a circle may be considered as a particular case of an ellipse, when the two axes are equal.* Therefore in drawing the graph of a circle or of an ellipse, the student should be particularly careful that *the same scale of units along both the axes is assumed,* for, if different scales of units are taken in the case of a circle the shape of the graph will be that of an ellipse (*i.e.*, oval-shaped), and in the case of an ellipse it may happen that the shape of the graph becomes that of a circle.

(*d*) *Hyperbola.*

Example 8. *Draw the graph of* $4x^2-9y^2=36$.

The given equation can be written in either of the forms

$$y=\pm\tfrac{2}{3}\sqrt{x^2-9} \qquad \ldots \qquad \ldots \qquad (1)$$

or,

$$x=\pm\tfrac{3}{2}\sqrt{y^2+4} \qquad \ldots \qquad \ldots \qquad (2)$$

From the form (1) let us tabulate the values of x and y satisfying the equation. Thus,

when	$x=\pm3$,	±4,	±5,	±6,	±7.
we have	$y=0$,	$\pm1{\cdot}8$,	$\pm2{\cdot}7$,	$\pm3{\cdot}6$,	$\pm4{\cdot}2$.

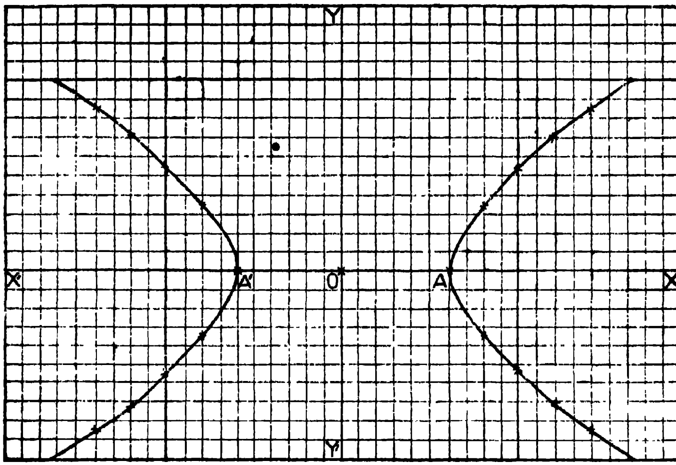

Fig. 10.

That is, the series of points (3, 0), (– 3, 0), (4, 1·8) (4, –1·8), (–4, 1·8), (–4, –1·8), etc., lie on the graph.

Let us take two small divisions for the unit of length and plot these points. Then the graph required is the curve drawn freehand through the points as shown in Fig. 10. A curve of this class is called a *Hyperbola.*

NOTE 1. As in the case of the ellipse, from the equations (1) and (2) above it is clear that the hyperbola is symmetrical about the axis of x as well as the axis of y. Again, from the equation (1), when x lies between –3 and 3, x^2 is less than 9. *i.e.*, x^2-9 is negative, and therefore y is imaginary. Hence there is no part of the curve lying between the lines $x=-3$ and $x=3$. Thus the curve consists of two branches, one lying entirely on the right side of $x=3$, and the other on the left side of $x=-3$. The *major axis* of the hyperbola is AOA' ; O is called the *centre*, and A and A' the *vertices* of the hyperbola, and the straight line through O perpendicular to AOA' is called the *transverse axis*.

NOTE 2. From the equation (1), as x increases either in the positive direction or in the negative direction, y also increases with x; and since there is no limit to this increase of x and y, both the branches of the curve extend to infinity, one towards the positive direction and the other towards the negative direction of the axis of x.

NOTE 3. If the equation given be $4x^2-9y^2=-36$, *i. e.*, $9y^2-4x^2=36$, its graph will also be a hyperbola consisting of two branches, one lying entirely above the line $y=2$, and the other below the line $y=-2$. There is no part of the curve which lies between $y=-2$ and $y=2$. The student should verify this by actually drawing the graph.

NOTE 4. Generally, the graph of an equation of the form $ax^2-by^2=\pm c$, in which a, b, c are all positive is a hyperbola. The equation of a hyperbola is usually written in the form

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

When $a=b$, the hyperbola is called a ***rectangular or equilateral hyperbola***, and its equation is

$$x^2-y^2=a^2.$$

Example 9. *Draw the graph of $xy=4$.*

From the given equation, we have $y=\frac{4}{x}$.

Hence, when $x=\frac{1}{2}$, 1, 2, 4, 8 we have $y=8, 4, 2, 1, \frac{1}{2}$; and when $x=-\frac{1}{2}, -1, -2, -4, -8$, $y=-8, -4, -2, -1, -\frac{1}{2}$.

Thus when x is positive, y is also positive, and when x is negative, y is negative ; hence the curve lies in the first and third quadrants only and no part of the curve lies either in the second or in the fourth quadrant.

Taking two small divisions for the unit of length and proceeding as usual, the required graph is the curve drawn in Fig. 11. It is a particular case of a hyperbola and is called a ***rectangular or equilateral hyperbola.***

NOTE 1. Since $xy=(-x)(-y)$, corresponding to any point (x,y) on the curve $xy=4$, there is another point $(-x, -y)$ on the same curve, one lying in the first and the other in the third quadrant. Hence the curve consists of two branches, in the first and the third quadrants, one being exactly similar to the other.

NOTE 2. From $y=\frac{4}{x}$, it is clear that as x increases either in the positive or in the negative direction, y numerically diminishes and ultimately when x is infinitely large, y becomes infinitely small; hence both the branches of the curve continually approach the x-axis but never meet it, for any finite value of x. But when $x=\infty$, we have $y=0$ and the curve meets the x-axis at infinity. Similarly,

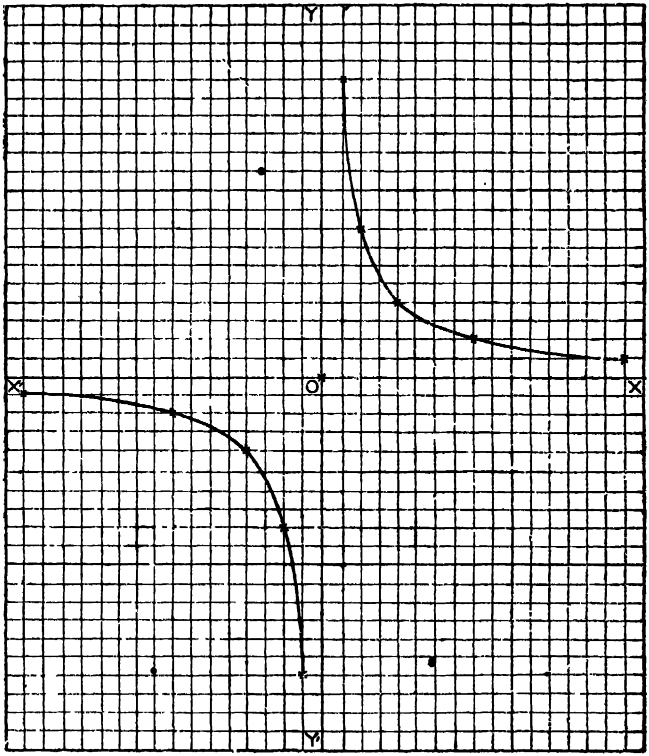

Fig. 11.

from $x = \frac{\quad}{y}$ it can be shown that both the branches of the curve continually approach the y-axis as y increases and ultimately when $y = \infty$, the curve meets the y-axis at infinity. The two axes so related to the curve are called its *asymptotes*.

NOTE 3. The graph of $xy = -4$ is also a rectangular hyperbola which consists of two branches, one lying in the second quadrant and the other in the fourth quadrant. The student should verify this by actually drawing the graph.

NOTE 4. From the above example it is evident that every equation of the form $xy = \pm c$ is a rectangular hyperbola.

107. The Five Cases and their Conditions.

In the preceding Articles we have seen that when an equation of the second degree in x and y is given, its graph is either (i) a pair of straight lines, or (ii) a circle, or (iii) a parabola, or (ix) an ellipse, or (v) a hyperbola.

When such an equation is given, the following conditions may be used by the student before beginning to draw the graph, as tests to know what sort of curve the equation will represent. The student must, however, take for granted the truth of the conditions here stated because their proof is beyond the scope of the present treatise.

Let us take $ax^2+2hxy+by^2+2gx+2fy+c=0$ for the general equation of the second degree. Then

(i) If the relation $abc+2fgh-af^2-bg^2-ch^2=0$ between the co-efficients holds, the quadratic expression can be resolved into two linear factors and the equation will represent a pair of straight lines. [See Art. 85.]

If the condition given in (i) does not hold, then the equation will represent

(ii) A *circle*, if $a=b$ and $h=0$, *i.e.*, if the co-efficients of x^2 and y^2 are equal and there is no term containing xy. Thus, the equations $x^2+y^2=16$, $3x^2+3y^2-5x+6y+2=0$ are all circles.

(iii) A *parabola*, if $h^2=ab$, *i.e.*, if the terms of the second degree constitute a perfect square. Thus, $y^2=2x+5$, $y=2x^2-5x+3$, $(x-y)^2-2(x+y)+1=0$ are all parabolas.

(iv) An *ellipse*, if $h^2<ab$, *i.e.*, if the terms of the second degree cannot be resolved into two real factors. Thus, $4x^2+9y^2=36$, $7x^2-3xy+11y^2=15$, $2(x-1)^2+3(y-2)^2=1$, are all ellipses.

(v) A *hyperbola*, if $h^2>ab$, *i.e.*, if the terms of the second degree can be resolved into two real factors. Thus, $x^2-y^2=1$, $2x^2-3y^2=12$, $(x+2y)(2x-3y)+4x-6y+5=0$ are all hyperbolas. If, moreover, $a+b=0$, the curve is a rectangular hyperbola.

EXERCISE 36.

Draw the graphs of the following equations :

1. $x^2+y^2=36$. 2. $x^2+y^2=58$. 3. $y^2=4x$.

4. $y^2+4a=0$. 5. $4x^2+9y^2=1$. 6. $4x^2-9y^2=1$.

7. $y=3x^2-18x+5$. **8.** $(x-4)^2+(y+5)^2=49$.

9. $(y+3)^2=2(x+5)$. **10.** $y=(x-3)(x-5)$.

11. $x^2+y^2-2x-4y-4=0$. **12.** $x^2+y^2+6x-14y+33=0$.

13. $4x^2+9y^2-8x-36y+4=0$. **14.** $y^2+2y+x+4=0$.

15. $9x^2-16y^2=144$. **16.** $xy=1$. **17.** $(x+1)(y+2)=30$.

18. $y=\dfrac{1}{x+1}$. **19.** $\dfrac{1}{x}+\dfrac{1}{y}=1$. **20.** $y=\dfrac{x-1}{x-2}$.

21. What do you understand by the graph of the equation $x^2-y^2=0$?

22. Show that the graph of the equation $2x^2+3y^2+6=0$ is impossible.

23. Draw the graph of $\sqrt{x}+\sqrt{y}=1$, and show that the curve touches both the axes at the points (1, 0) and (0, 1) respectively.

24. Draw the graph of $x^2+y^2=6x$, and show that the axis of y touches the curve at the origin.

25. Find the equation of the circle whose centre is the point (3, 4) and radius equal to 5.

26. Trace the graph of $y=x^2-4x+5$ from $x=0$ to $x=4$, and find the least value of y. [Cal. 1918]

27. Draw the graph of $x^2=13$ by a practical method, without extracting the square root of 13.

[*Hints*: Plot a point $P(2, 3)$ and join OP, O being the origin; then $OP^2=2^2+3^2=13$, and hence, $OP=\sqrt{13}$. With centre O and radius OP describe a circle which cuts x-axis at M, N; then OM and ON are equal to $+\sqrt{13}$ and $-\sqrt{13}$ respectively.]

28. Draw the graph of $y^2=34$ by a practical method, without extracting the square root of 34.

29. Find graphically the square roots of 2, 3, 5 and 7 to one place of decimal.

30. Draw to a large scale the graph of $y=x^2$ from $x=6$ to $x=7$; from the graph find, as accurately as your scales allow, $\sqrt{45}$. [Pat. 1918]

108. Graphical Solution of Quadratic Equations.

The following examples will illustrate the method :

Example 1. *Solve graphically the equation* $2x^2-8x+7=0$; (*i*) *by the one-graph method*, (*ii*) *by the two-graph method.*

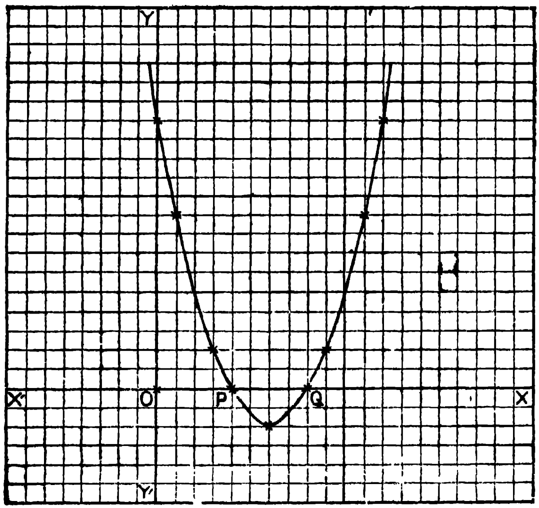

Fig. 12.

(i) Let us draw the graph of the equation $y=2x^2-8x+7$, then the abscissæ of the points at which the graph cuts the x-axis are the roots required, for at those points $y=0$.

From the equation $y=2x^2-8x+7$,

when	$x=0$,	$\frac{1}{3}$,	1,	2,	3,	$3\frac{2}{3}$,	4,
we have	$y=7$,	$4\frac{5}{9}$,	1,	-1,	1,	$4\frac{5}{9}$,	7.

Taking 3 small divisions = unity along OX, and 2 small divisions = unity along OY, and proceeding in the usual way, we find the graph to be a parabola, as shown in Fig. 12. The graph cuts the axis of x at P and Q, where $OP=4$ and $OQ=8$ small divisions nearly. Hence, since 3 small divisions = unity along OX, the two roots of the equation $2x^2-8x+7=0$ are $\frac{4}{3}$ and $\frac{8}{3}$ *i.e.*, 1·3 and 2·7 nearly.

(ii) Writing the equation in the form $x^2=4x-\frac{7}{2}$, if we draw the graphs of $y=x^2$ and $y=4x-\frac{7}{2}$ on the same diagram, using the same scales of units, then it is evident that the abscissæ of the points at which the parabola $y=x^2$ is cut by the straight line $y=4x-\frac{7}{2}$, will be the roots of the given equation $2x^2-8x+7=0$.

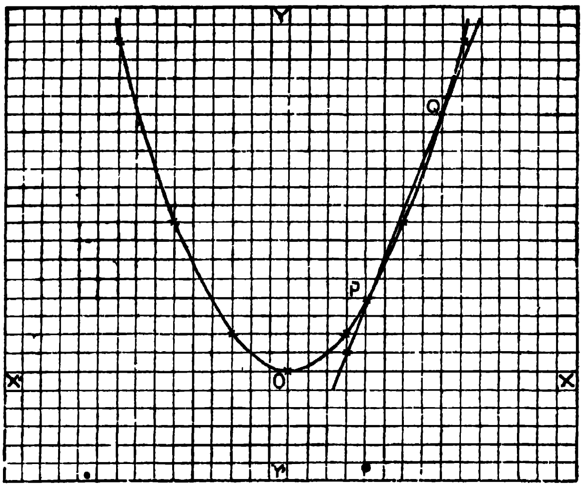

Fig. 13

From the equation $y=x^2$,

when	$x=0,$	$\pm 1,$	$\pm 2,$	$\pm 3,$	...
we have	$y=0,$	$1,$	$4,$	$9,$	...

and from the equation $y=4x-\frac{7}{2}$, we see that the points $(1, \frac{1}{2})$ and $(3, \frac{17}{2})$ are on the graph. Taking 3 small divisions for the unit along OX and 2 small divisions for the unit along OY, we get the two graphs, one a parabola and the other a straight line, as in Fig. 13, intersecting at P and Q. We find that the abscissæ of P and Q are 4 and 8 small divisions nearly, and therefore the roots of $2x^2-8x+7=0$ are $\frac{4}{3}$ and $\frac{8}{3}$, *i.e.*, 1·3 and 2·7 nearly.

Note 1. From the above example it is clear that since every quadratic equation can always be reduced to the form $ax^2+bx+c=0$.

we can solve the equation either (i) by using one graph, or (ii) by using two graphs.

In the former case, we draw the graph of $y=ax^2+bx+c$ (which is evidently a parabola), and the roots of $ax^2+bx+c=0$ are the values of x where the graph cuts the axis of x. If however, the graph does not cut the axis of x, the roots cannot be found out by the graphical method, and in this case the roots are *imaginary.*

In the latter case, the best method is to arrange the equation in the form $x^2=-\frac{b}{a}x-\frac{c}{a}$, so that only x^2 is on one side of the equation and the terms of lower degree on the other side ; then on the same diagram, draw the graphs of $y=x^2$ (which is a parabola), and $y=-\frac{b}{a}x-\frac{c}{a}$ (which is a straight line), and the two roots of the equation are the abscissæ of the points, at which the two graphs intersect each other. It must however be observed that the two roots will be *real and unequal*, or *real and equal*, or *imaginary*, according as the line $y=-\frac{b}{a}x-\frac{c}{a}$ cuts the parabola $y=x^2$ in two distinct points, or in two coincident points, or in no point. In the last case, *i.e.*, of *imaginary* roots, it is evident that the graphical method fails to solve the equation. Thus, all quadratic equations with real roots can be solved graphically by using the parabola $y=x^2$ which is the same and serves for all quadratics, and a straight line the equation of which is of course deduced from the equation to be solved and is different for different equations.

NOTE 2. We may also solve a quadratic equation by using different pairs of graphs, their equations being deduced from the given quadratic by transposition. Thus by transposition, we can write the equation $2x^2-8x+7=0$ of Example 1, in the form (i) $2x^2-8x=-7$, or (ii) $x^2-4x+3=-x^2+4x-4$, or (iii) $2x^2=8x-7$, etc. and the two roots of the quadratic are given by the abscissæ of the points at which (i) the line $y=-7$ intersects the parabola $y=2x^2-8x$, or (ii) the parabola $y=x^2-4x+3$ intersects the parabola $y=-x^2+4x-4$. or (iii) the line $y=8x-7$ intersects the parabola $y=2x^2$; and so on.

Example 2. *Solve graphically the equations :*

$$\left.\begin{array}{lr} x^2+y^2=25 \quad \cdots \quad \cdots \quad \cdots & (1) \\ 3x-2y=\ 1 \quad \cdots \quad \cdots \quad \cdots & (2) \end{array}\right\}$$

The graph of the equation (1) is a circle whose centre is the origin and radius equal to 5 units : and the graph of the equation (2) is the straight line joining the points $(-3,-5)$ and $(5, 7)$. These two graphs intersect

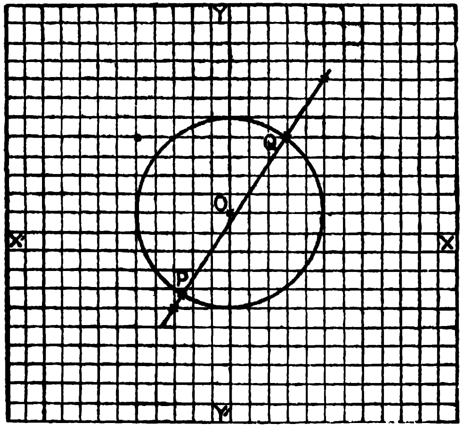

Fig. 14

at P and Q as shown in Fig. 14, whose co-ordinates are $(-2{\cdot}5, -4{\cdot}3)$ and $(3, 4)$ respectively. Hence, the required roots are either $x = -2{\cdot}5$ and $y = -4{\cdot}3$, or $x = 3$ and $y = 4$.

EXERCISE 37.

1. Solve the following equations graphically (i) by the one-graph method, and (ii) by the two-graph method :

(i) $x^2 + x - 2 = 0$. (ii) $x^2 - x - 6 = 0$.

(iii) $2x^2 - 7x + 3 = 0$. (iv) $2x^2 - 3x = 0$.

(v) $4x^2 + 12x + 9 = 0$. (vi) $6x^2 + x - 1 = 0$.

(vii) $3x^2 + 7x - 6 = 0$. (viii) $x^2 - 3x - 2 = 0$.

(ix) $2x^2 - 5x - 1 = 0$. (x) $2x^2 - 3x - 4 = 0$.

2. Solve the following equations graphically :

(i) $x^2 + y^2 = 25,\ x + y = 1$. (ii) $x^2 + y^2 = 169,\ x + y = 7$

(iii) $x^2 + y^2 = 16,\ 2x + 3y = 6$. (iv) $y = 3x^2,\ y = 2x + 1$.

(v) $y^2 = 5x,\ y = 2x$. (vi) $x + y = 5,\ xy = 6$.

(vii) $x^2 - y^2 = 16,\ x + y = 8$. (viii) $4x^2 + 9y^2 = 25,\ xy = 2$.

(ix) $x^2 + 4y^2 = 34,\ x + 4y = 13$.

(x) $25x^2 + 9y^2 = 181,\ x^2 + y^2 = 13$.

3. Draw the graph of x^2-5x, and hence find the roots of the equation $x^2-5x+6=0$.

4. Draw the graph of x^2-2x, and hence find the roots of the equation $x^2-2x=3$.

5. Trace the graph of $y=x^2-x$ from $x=-1$ to $x=2$, and thence obtain an approximate solution of the equation $1=x^2-x$. [Cal. 1917]

6. Draw the graphs of (i) $x^2+y^2=36$, and (ii) $3x+4y=30$. Prove that the second graph touches the first and find the co-ordinates of the point of contact.

7. Find graphically the minimum value of the expression $2x^2-10x+7$.

8. Find graphically the maximum value of the expression $5+12x-3x^2$.

9. Trace the graphs of the following equations, and determine their points of intersection : $x-y+1=0$, $y=2x^2$. [Pat. 1919.]

10. Draw the graphs of $x^2+y^2=16$ and $x+y=2$, and measure the length of the chord of intersection. [Cal. 1913.]

11. In the same diagram, draw the graphs of $2x+1$ and x^2. From your graphs read, as accurately as you can, the value or values of x which will make $x^2=2x+1$. [Cal. 1930.]

12. Draw, choosing your own scale, between $x=\pm 4$, the following graphs, $y=\frac{1}{2}x^2$, $y=2x-2$, $4y+5x=-3$, $y=x-2$; and state what you can infer from your graphs about the roots of the equations $\frac{1}{2}x^2=2x-2$, $2x^2+5x+3=0$, and $x^2=2x-4$. [All. 1919]

CHAPTER XI

IMAGINARY QUANTITIES.

109. Imaginary Quantities.

Since any real quantity, positive or negative, when squared gives rise to a positive quantity, it is evident that a negative quantity cannot have any real square root. In course of algebraic operations, however, we do meet with square roots of negative quantities. For instance, in solving the quadratic equation $ax^2+bx+c=0$, if $b^2<4ac$, the expression $\sqrt{b^2-4ac}$ that occurs is the square root of a negative quantity. Since no real quantity can be equivalent to such expressions, these are called *imaginary (or impossible) quantities.* [Art. 8.]

Thus $\sqrt{-1}$, $\sqrt{-2}$, $\sqrt{-a^2}$, $\sqrt{-a^4}$,.......are imaginary quantities,

It is not merely the square root of a negative quantity that gives rise to imaginary quantities; but any even root of a negative quantity would give rise to such quantities for any real quantity, positive or negative, raised to any even power would give a positive result.

Thus $\sqrt[4]{-1}$, $\sqrt[4]{-2}$, $\sqrt[6]{-3}$, $\sqrt[6]{-5}$,.........are all imaginary quantities.

110. Fundamental Laws of Algebra applicable to Imaginaries.

Though as explained above, the square root of a negative quantity cannot be a real number, still, since such expressions do occur in Algebra, it is desirable to find, if possible, some meaning for them, that is to say, to frame definitions for them and to assign to them properties and laws of combination consistent, as far as possible, with the fundamental laws and processes of Algebra satisfied by real quantities.

Accordingly, if a is positive quantity, we *define* $\sqrt{-a}$ as the quantity which being multiplied by itself or squared becomes equal to $-a$. That is to say,

$$\sqrt{-a} \times \sqrt{-a} = (\sqrt{-a})^2 = -a.$$

Also, the Associative, Commutative and Distributive Laws, and relations like $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$, which are true when $\sqrt{a}$ and $\sqrt{b}$ are real quantities, are assumed to hold good when the quantities are imaginary. Hence.

$$\sqrt{-a} = \sqrt{a(-1)} = \sqrt{a}.\sqrt{-1}$$

$$\sqrt{-a} + \sqrt{-b} + \sqrt{-c} = \sqrt{a}.\sqrt{-1} + \sqrt{b}.\sqrt{-1} + \sqrt{c}.\sqrt{-1}$$
$$= (\sqrt{a} + \sqrt{b} + \sqrt{c})\sqrt{-1}.$$

$$\sqrt{-a}.\sqrt{-b} = \sqrt{a(-1)}\sqrt{b(-1)} = \sqrt{a}.\sqrt{-1}.\sqrt{b}.\sqrt{-1}$$
$$= \sqrt{a}.\sqrt{b}.\sqrt{-1}\sqrt{-1} = \sqrt{ab}(\sqrt{-1})^2 = -\sqrt{ab}.$$

$$\sqrt[4]{-1} = \sqrt{(\sqrt{-1})} = (\sqrt{-1})^{\frac{1}{2}}.$$

111. Square root of -1 [*i.e.*, $\sqrt{-1}$ or i].

It follows that all imaginary quantities can be ultimately reduced to $\sqrt{-1}$, or some fractional power of $\sqrt{-1}$.

The square root of -1, *i.e.*, $\sqrt{-1}$, may therefore be looked upon as the fundamental imaginary quantity. This quantity $\sqrt{-1}$ is usually denoted by the letter i (the first letter of the word *imaginary*).

Example 1. *What is the fallacy in the following arguments* :

(i) $\sqrt{-1} \times \sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{(-1)^2} = \sqrt{+1} = \pm 1$;

(ii) $\sqrt{-a} \times \sqrt{-b} = \sqrt{(-a)(-b)} = \sqrt{ab}$?

(i) The fallacy lies in the last step, when for $\sqrt{+1}$ we write ± 1. The reason is this. When we generally take a square root of a certain quantity it is unknown which particular root squared has produced the given quantity, and therefore we place the double sign $\pm$ 1 to include both the possible cases. But if it is known that a particular root squared has produced the given quantity, then the proper sign to take in that case is that which gives that particular square root. Here the previous step $\sqrt{(-1)^2} = \sqrt{+1}$ shows that $+1$ has been obtained by squaring -1 ; hence in this case the proper square root is -1. Hence $\sqrt{-1} \times \sqrt{-1} = -1$.

(ii) Here also, the fallacy lies in the last step. In fact, the last step conceals four other steps, *viz.*

$$\sqrt{(-a)(-b)}=\sqrt{a(-1).b(-1)}=\sqrt{ab}.\sqrt{(-1)^2}=\sqrt{ab}.\sqrt{1}=\sqrt{ab}.$$

In the given argument it thus appears that $\sqrt{(-1)^2}$ has been tacitly taken to be equal to $+1$, which is wrong from what has been said above. The proper argument should be :

$$\sqrt{-a}\times\sqrt{-b}=\sqrt{(-a)(-b)}=\sqrt{a.(-1).b(-1)}$$
$$=\sqrt{ab}\sqrt{(-1)^2}=\sqrt{ab}.(-1)=-\sqrt{ab}.$$

112. Powers of i.

From what has been stated above it is evident that the successive positive integral powers of i or $\sqrt{-1}$ are as follows :

$i^1=i$; $\quad i^2=(\sqrt{-1})^2=-1.$

$i^3=i^2.\, i=(-1)i=-i$; $\quad i^4=(i^2)^2=(-1)^2=1.$

$i^5=i^4.\, i=1.i=i$; $\quad i^6=i^4.i^2=1.(-1)=-1.$

$i^7=i^4.\, i^3=1.(-i)=-i$; $\quad i^8=(i^4)^2=1^2=1$; etc.

Generally we have, if n is an integer,

$$i^{4n}=(i^4)^n=1^n=1$$
$$i^{4n+1}=i^{4n}.i=1.i=i$$
$$i^{4n+2}=i^{4n}.i^2=1.(-1)=-1$$
$$i^{4n+3}=i^{4n}.i^3=1.(-i)=-i.$$

Thus, the positive integral powers of i admit of only four values, of which two are real, *viz.*, ± 1, and two imaginary, *viz.*, $\pm i$.

Negative powers of i give the following results :

$$i^{-1}=\frac{1}{i}=-\frac{(-1)}{\sqrt{-1}}=-\sqrt{-1}=-i\ ;\quad i^{-2}=\frac{1}{i^2}=\frac{1}{-1}=-1\ ;$$

$$i^{-3}=\frac{1}{i^3}=\frac{1}{-i}=i\ ;\qquad i^{-4}=\frac{1}{i^4}=\frac{1}{1}=1\ ;$$

and so on ; so that the negative integral powers of i also give the same four values ± 1 and $\pm i$ over again.

113. The Imaginary as an Operator.

A very interesting interpretation of the imaginary quantity $\sqrt{-1}$ may be obtained in the following manner.

If a certain length in one direction is taken as positive, then an equal length in the opposite direction is denoted by putting a $-$ sign to the measure of the length; so that multiplication by -1 may be looked upon as an *operator* which turns the given length through two right angles.

Now, since by definition,

$$\sqrt{-1}.\sqrt{-1}=-1,$$

it appears that $\sqrt{-1}$ *may be regarded as an operator which turns through one right angle*; because another repetition of the same operation rotates through two right angles, which is indicated by -1. Similarly, four such operations would rotate the given line through four right angles and bring it back to the original position. Hence, $(\sqrt{-1})^4=1$.

EXERCISE 38.

1. Find the values of :

(i) $3\sqrt{-4}+5\sqrt{-9}$. (ii) $7\sqrt{-16}-5\sqrt{-25}$.

(iii) $5\sqrt{-3}\times 6\sqrt{-5}$. (iv) $2\sqrt{-2}\times 3\sqrt{-3}\times 4\sqrt{-4}$.

(v) $6\sqrt{-12}\div 2\sqrt{-3}$. (vi) $-\sqrt{-4}\div 6\sqrt{-3}$.

(vii) $\{3\sqrt{-6}\}^2$. (viii) $\{2\sqrt{-18}\}^2\times\{3\sqrt{-3}\}^4$.

2. Find the values of :

(i) i^{21}. (ii) i^{38}. (iii) i^{51}. (iv) i^{60}.

(v) i^{-13}. (vi) i^{-22}. (vii) i^{-39}. (viii) i^{-48}.

114. Complex Numbers.

Any expression of the form $a+bi$, where a and b are real quantities, is called an *Imaginary Expression* or a *Complex Number*. It consists of a real part a and an imaginary part bi. Thus, $2+3\sqrt{-1}$, $3-5\sqrt{-1}$, and $1+\sqrt{-3}$ are all imaginary expressions or complex numbers.

If $b=0$, the imaginary part vanishes, and the complex number reduces to a real one.

115. Theorems concerning Complex Numbers.

(a) *If $a+bi=0$, then $a=0$, $b=0$.*

Since $a+bi=0$, we have $a=-bi$, that is, a real quantity is equal to an imaginary quantity, which is impossible unless each of them is zero.

$$\therefore \qquad a=0, \text{ and } b=0.$$

(b) *If $a+bi=x+yi$, then $a=x$ and $b=y$.*

Since $a+bi=x+yi$, we have, by transposition.

$$(a-x)+(b-y)i=0.$$

Hence from Theorem (a), we have $a-x=0$ and $b-y=0$; that is, $a=x$, and $b=y$.

That is to say, *when two complex numbers are equal to one another, the real part is equal to the real part, and the imaginary part to the imaginary.*

(c) *The algebraic sum of any number of complex numbers is a complex number.*

Let $a+bi$, $c+di$, $e+fi$, etc. be the complex numbers, then we have

$$(a+bi)-(c+di)+(e+fi)+\cdots\cdots$$
$$=(a-c+e+\cdots\cdots)+(b-d+f+\cdots\cdots)i,$$

which is of the form $\alpha+\beta i$, and therefore is a complex number. Of course if $b-d+f+\cdots=0$, the algebraic sum reduces to a real quantity.

(d) *The product of any number of complex numbers is a complex number.*

Let $a+bi$, $c+di$, $e+fi$, etc. be the complex numbers, then we have by actual multiplication

$$(a+bi)(c+di)=(ac-bd)+(ad+bc)i \qquad \dots \text{(i)}$$

$$(a+bi)(c+di)(e+fi)=\{(ac-bd)+(ad+bc)i\}(e+fi)$$
$$=\{(ac-bd)e-(ad+bc)f\}+\{(ac-bd)f+(ad+bc)e\}i \qquad \text{(ii)}$$

and so on.

The results (i) and (ii) are evidently of the form $\alpha+\beta i$, and are therefore complex numbers. By proceeding in this way it can be shown that the continued product of any number of complex numbers is a complex number. Here also, in case the co-efficient of the imaginary part vanishes, the product becomes a real quantity.

(*e*) *The quotient obtained by dividing one complex number by another is a complex number.*

Let $a+bi$ and $c+di$ be any two complex numbers.

Then, $\frac{a+bi}{c+di}=\frac{(a+bi)(c-di)}{(c+di)(c-di)}=\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$

$$=\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}.i,$$

a complex number. Here also, if $bc=ad$, the quotient becomes real.

(*f*) *Any power of a complex number is a complex number.*

Let $a+bi$ be a complex number; then

$(a+bi)^2=(a^2-b^2)+2abi$

$(a+bi)^3=(a^3-3ab^2)+(3a^2b-b^3)i,$

$(a+bi)^4=(a^4-6a^2b^2+b^4)+4ab(a^2-b^2)i$, and so on, which are all of the form $\alpha+\beta i$. By proceeding in this way, any power of $a+bi$ may be shown to be a complex number.

(*g*) *Any root of a complex number is a complex number.*

Let $a+bi$ be a complex number, and let $\sqrt[n]{a+bi}=x$.

Then, $$x^n=a+bi.$$

Now, if x be real, then x^n must be real; but in this case $x^n=a+bi$, an imaginary expression, and therefore x cannot be real; hence x must be a complex number.

116. Square Root of a Complex Number.

Since any root of a complex number is a complex number, we assume

$$\sqrt{a+bi}=x+yi,$$

where x and y are real quantities.

Then, $$a+bi=(x+yi)^2=x^2-y^2+2xyi.$$

Hence, equating the real and imaginary parts separately, we have

$$\left.\begin{aligned} x^2-y^2&=a \quad \cdots \quad \cdots \quad (1)\\ 2xy&=b \quad \cdots \quad \cdots \quad (2)\end{aligned}\right\}$$

Squaring (1) and (2) and adding,

$$(x^2-y^2)^2+4x^2y^2=a^2+b^2$$

or, $$(x^2+y^2)^2=a^2+b^2$$

$$\therefore x^2+y^2=\sqrt{a^2+b^2} \quad \cdots \quad \cdots \quad (3)$$

Since x and y are real quantities, x^2+y^2 is positive, and therefore in (3) we take the positive value of the quantity $\sqrt{a^2+b^2}$.

From (1) and (3), we have

$$x^2=\frac{\sqrt{a^2+b^2}+a}{2} \text{ and } y^2=\frac{\sqrt{a^2+b^2}-a}{2}$$

$$\therefore x=\pm\left\{\frac{\sqrt{a^2+b^2}+a}{2}\right\}^{\frac{1}{2}} \text{ and } y=\pm\left\{\frac{\sqrt{a^2+b^2}-a}{2}\right\}^{\frac{1}{2}}.$$

Now from (2) we see that the product xy must have the same sign as b, that is,

(i) if b be positive, xy must be positive, and therefore x and y must be either both positive or both negative; and

(ii) if b be negative, xy must be negative, and therefore one of the quantities x and y must be positive and the other negative.

Hence when b is positive, the required square roots are

$$\pm\left[\left\{\frac{\sqrt{a^2+b^2}+a}{2}\right\}^{\frac{1}{2}}+\left\{\frac{\sqrt{a^2+b^2}-a}{2}\right\}^{\frac{1}{2}}i\right]$$

and when b is negative, the required square roots are

$$\pm\left[\left\{\frac{\sqrt{a^2+b^2}+a}{2}\right\}^{\frac{1}{2}}-\left\{\frac{\sqrt{a^2+b^2}-a}{2}\right\}^{\frac{1}{2}}i\right].$$

Cor. If $a=0$, and $b=\pm1$, we have

$$\sqrt{+i}=\pm\frac{1+i}{\sqrt{2}} \text{ and } \sqrt{-i}=\pm\frac{1-i}{\sqrt{2}}.$$

These results may also be obtained directly, thus:

Since $\qquad i=\frac{1}{2}(1+2i-1)=\frac{1}{2}(1+2i+i^2)=\frac{1}{2}(1+i)^2$

$$\therefore \qquad \sqrt{i}=\pm\frac{1+i}{\sqrt{2}}.$$

Again, since $-i=\frac{1}{2}(1-2i-1)=\frac{1}{2}(1-2i+i^2)=\frac{1}{2}(1-i)^2$

$$\therefore \qquad \sqrt{-i}=\pm\frac{1-i}{\sqrt{2}}$$

Example 1. *Write down the product of* $2+3i$ *and* $5-2i$.

$$(2+3i)(5-2i)=10+15i-4i+6$$
$$=16+11i.$$

Example 2. *Write down the square of* $\frac{1}{\sqrt{2}}\{(\sqrt{2}+1)^{\frac{1}{2}}+(\sqrt{2}-1)^{\frac{1}{2}}i\}$.

The square of the given expression

$$=\tfrac{1}{2}\{(\sqrt{2}+1)^{\frac{1}{2}}+(\sqrt{2}-1)^{\frac{1}{2}}i\}^2$$

$$=\tfrac{1}{2}\{(\sqrt{2}+1)-(\sqrt{2}-1)+2(\sqrt{2}+1)^{\frac{1}{2}}(\sqrt{2}-1)^{\frac{1}{2}}i\}.$$

$$=\tfrac{1}{2}\{2+2i\}$$

$$=1+i.$$

Example 3. *Find the square root of* $7-24i$.

Let $\sqrt{7-24i}=x+yi$; then $7-24i=(x+yi)^2=x^2-y^2+2xyi$.

$$\therefore \quad \left.\begin{aligned} x^2-y^2&=7 \quad \ldots \quad \ldots \quad (1)\\ 2xy&=-24 \quad \ldots \quad \ldots \quad (2)\end{aligned}\right\}$$

Squaring (1) and (2) and adding,

$$(x^2-y^2)^2+4x^2y^2=7^2+24^2$$

or, $$(x^2+y^2)^2=49+576=625$$

$$\therefore \quad x^2+y^2=25 \quad \ldots \quad \ldots \quad (3)$$

From (1) and (3), $x^2=16$ and $y^2=9$

$$\therefore \quad x=\pm 4 \text{ and } y=\pm 3.$$

From (2), since the product $xy=-12$, a negative quantity, we must have $x=4$, $y=-3$, or, $x=-4$ and $y=3$; and the required square roots therefore are $4-3i$ and $-4+3i$, that is, $\pm(4-3i)$.

Example 4. *Find the square root of* $2a+(a^2-1)i$.

Let $\sqrt{\{2a+(a^2-1)i\}}=x+yi$

$\therefore\ 2a+(a^2-1)i=(x+yi)^2=x^2-y^2+2xyi$

Then, $$x^2-y^2=2a \quad \ldots \quad \ldots \quad (1)$$

$$2xy=a^2-1 \quad \ldots \quad \ldots \quad (2)$$

From (1) and (2), $(x^2+y^2)^2=(2a)^2+(a^2-1)^2=(a^2+1)^2$.

$$\therefore \quad x^2+y^2=a^2+1 \quad \ldots \quad \ldots \quad (3)$$

From (1) and (3), $x^2=\frac{1}{2}(a^2+2a+1)=\frac{1}{2}(a+1)^2$

and $y^2=\frac{1}{2}(a^2-2a+1)=\frac{1}{2}(a-1)^2$

$$\therefore \quad x=\pm\frac{1}{\sqrt{2}}(a+1),\ y=\pm\frac{1}{\sqrt{2}}(a-1).$$

$\therefore$ the required square roots are $\pm\frac{1}{\sqrt{2}}\{(a+1)+(a-1)i\}$.

117. Conjugate Complex Numbers.

When two complex numbers are such that they differ only in the sign of the co-efficient of i, they are said to be *conjugate complex numbers*. Thus, $a+bi$ and $a-bi$ are conjugate complex numbers.

Since $\qquad (a+bi)+(a-bi)=2a,$
and $\qquad (a+bi)(a-bi)=(a^2+b^2),$

it at once follows that *the sum and the product of two conjugate complex numbers are real.*

Example 1. *Express* $\dfrac{13+41i}{4+3i}$ *in the form* $\alpha+\beta i$.

Multiplying both the numerator and the denominator by the conjugate of the latter, we have

$$\frac{13+41i}{4+3i}=\frac{(13+41i)(4-3i)}{(4+3i)(4-3i)}=\frac{52+164i-39i+123}{16+9}=\frac{175+125i}{25}=7+5i.$$

118. Modulus of a Complex Number.

The positive value of the square root of a^2+b^2 is called the *modulus* of each of the complex numbers $a+bi$ and $a-bi$. Thus, the modulus of either of the expressions $2+3i$ and $2-3i$ is $\sqrt{(2^2+3^2)}$ *i.e.*, $\sqrt{13}$.

119. Properties of the Modulus.

(*i*) *The modulus of the product of two complex numbers is equal to the product of the moduli.*

Let $a+bi$ and $c+di$ be any two complex numbers so that their moduli are $\sqrt{a^2+b^2}$ and $\sqrt{c^2+d^2}$ respectively.

Then $\qquad (a+bi)(c+di)=(ac-bd)+(ad+bc)i.$

$$\therefore \text{ the modulus of the product} =\sqrt{(ac-bd)^2+(ad+bc)^2}$$
$$=\sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2}$$
$$=\sqrt{(a^2+b^2)(c^2+d^2)}$$
$$=\sqrt{a^2+b^2}\times\sqrt{c^2+d^2}$$

which proves the proposition.

(*ii*) *The modulus of the quotient of two complex numbers is the quotient of their moduli.*

Let $a+bi$ and $c+di$ be any two complex numbers.

Then $$\frac{a+bi}{c+di}=\frac{(a+bi)(c-di)}{(c+di)(c-di)}=\frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$
$$=\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}\cdot i.$$

the modulus of the quotient

$$=\sqrt{\left\{\left(\frac{ac+bd}{c^2+d^2}\right)^2+\left(\frac{bc-ad}{c^2+d^2}\right)^2\right\}}$$
$$=\sqrt{\left\{\frac{a^2c^2+b^2d^2+b^2c^2+a^2d^2}{(c^2+d^2)^2}\right\}}$$

$$=\sqrt{\left\{\frac{(a^2+b^2)(c^2+d^2)}{(c^2+d^2)^2}\right\}}$$

$$=\frac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}},$$

which proves the proposition.

Example 1. *Find the modulus of* $\dfrac{15+8i}{(4-3i)(12-5i)}$.

The required modulus $=\dfrac{\sqrt{(15^2+8^2)}}{\sqrt{(4^2+3^2)}.\sqrt{(12^2+5^2)}}=\dfrac{17}{5\times 13}=\dfrac{17}{65}$.

Example 2. *Simplify* $\dfrac{1+i}{1+2i}+\dfrac{1-i}{1-2i}$.

The given expression $=\dfrac{(1+i)(1-2i)+(1-i)(1+2i)}{(1+2i)(1-2i)}$

$$=\frac{(1-i-2i^2)+(1+i-2i^2)}{1-(2i)^2}$$

$$=\frac{2-4i^2}{1+4}=\frac{2+4}{5}=\frac{6}{5}.$$

Example 3. *Express* $(x^2+a^2)(x^2+b^2)(x^2+c^2)$ *as the sum of two squares.*

By actual multiplication, we have

$$\begin{aligned}(x+ai)(x+bi)(x+ci)&=x^3+(a+b+c)x^2i+(bc+ca+ab)xi^2+abci^3\\&=x^3+(a+b+c)x^2i-(bc+ca+ab)x-abci\\&=\{x^3-(bc+ca+ab)x\}+\{(a+b+c)x^2-abc\}i \quad (1)\end{aligned}$$

and $(x-ai)(x-bi)(x-ci)$

$$\begin{aligned}&=x^3-(a+b+c)x^2i+(bc+ca+ab)xi^2-abci^3\\&=\{x^3-(bc+ca+ab)x\}-\{(a+b+c)x^2-abc\}i \quad (2)\end{aligned}$$

Hence, multiplying (1) and (2), we have

$$(x^2+a^2)(x^2+b^2)(x^2+c^2)=\{x^3-(bc+ca+ab)x\}^2+\{(a+b+c)x^2-abc\}^2$$

Example 4. *Find the value of*

$$\sqrt{[-2+2\sqrt{(-2+2\sqrt{(-2+\dots ad\ inf.)})}]}.$$

Let x denote the given expression. Then we have

$$\begin{aligned}x^2&=-2+2\sqrt{\{-2+2\sqrt{(-2+\cdots ad.\ inf.)}\}}\\&=-2+2x\end{aligned}$$

or, $\qquad x^2-2x+2=0$

whence $x=\dfrac{2\pm\sqrt{4-8}}{2}=\dfrac{2\pm2\sqrt{-1}}{2}=1\pm\sqrt{-1}=1\pm i.$

EXERCISE 39.

1. Express the following in the form $A+Bi$:

(i) $\frac{22+7i}{5-4i}$. (ii) $\frac{1+6i}{7+5i}$. (iii) $\frac{9+19i}{5+3i}-\frac{13(2i-1)}{4+7i}$.

(iv) $\frac{a+bi}{a-bi}-\frac{a-bi}{a+bi}$. (v) $\left(\frac{a+bi}{a-bi}\right)^2-\left(\frac{a-bi}{a+bi}\right)^2$

2. Find the moduli of the following expressions :

(i) $1+i$. (ii) $3+4i$. (iii) $\frac{12+5i}{24+7i}$.

(iv) $\frac{(2+3i)(3-2\sqrt{2}i)}{(5+i)(5-3i)}$. (v) $(x+yi)^n$.

3. Multiply : (i) $7+4\sqrt{2}.i$ by $4-\sqrt{2}.i$.
(ii) $3\sqrt{-5}+5\sqrt{-3}$ by $3\sqrt{-3}-5\sqrt{-5}$.
(iii) $3\sqrt{-2}+2\sqrt{-5}$ by $4\sqrt{-2}-3\sqrt{-5}$.

4. Divide : (i) $3\sqrt{-2}+2\sqrt{-3}$ by $3\sqrt{-2}-2\sqrt{-3}$.
(ii) $6\sqrt{-1}-1$ by $5+7\sqrt{-1}$.

5. Express $(2+i)(3+2i)(4+3i)$ in the form $A+Bi$.

6. Express the continued product of $b+c-ai$, $c+a-bi$, and $a+b-ci$ in the form $A+Bi$.

7. Find the square roots of :

(i) $3+4i$. (ii) $8-15i$. (iii) $6\sqrt{-2}-7$.
(iv) $2\sqrt{-2}-1$. (v) $2a+2\sqrt{a^2+b^2}$. (vi) $1+i\sqrt{x^4-1}$
(vii) $1-2x\sqrt{(x^2-1)}.i$. (viii) $x+i\sqrt{x^4+x^2+1}$.

[Mad. F. A. 1888]

8. Extract the square root of $\frac{x^2}{y^2}+\frac{y^2}{x^2}+i\left(\frac{x}{y}-\frac{y}{x}\right)-\frac{9}{4}$

9. Find the fourth root of
$m^2(m^2-3n^2)+n^2(n^2-3m^2)+4mn(m-n)(m+n)i$.

10. Simplify $\{(\sqrt{2}+1)+(\sqrt{2}-1)i\}\{(\sqrt{2}-1)+(\sqrt{2}+1)i\}$.

11. Show that $\{(\sqrt{3}+1)+(\sqrt{3}-1)i\}^3=16(1+i)$.

12. If $x=\frac{a+bi}{c+di}$, show that

$$(c^2+d^2)x^2-2(ac+bd)x+(a^2+b^2)=0.$$

13. Show that
$(1+a^2)(1+b^2)(1+c^2)=(1-bc-ca-ab)^2+(a+b+c-abc)^2.$

14. Show that $(a^2+b^2\sqrt{-1})(a^2-b^2\sqrt{-1})$
$$=(a^2+ab\sqrt{2}+b^2)(a^2-ab\sqrt{2}+b^2).$$

15. Prove that $\sqrt{i}+\sqrt{-i}=\sqrt{2}$.

16. Show that
$$(1+i)^{\frac{1}{2}}=\frac{1}{\sqrt[4]{2}}\left\{\left(1+\frac{1}{\sqrt{2}}\right)^{\frac{1}{2}}+\left(1-\frac{1}{\sqrt{2}}\right)^{\frac{1}{2}}i\right\}.$$

17. Find the value of
$$x^4+4x^3+6x^2+4x+9 \text{ when } x=\sqrt{-2}-1.$$
[Cal. F. A. 1886]

18. Find the value of x^3+7x^2-x+16 when $x=1+2\sqrt{-1}$.
[All. I. E. 1926]

19. Express $\dfrac{m(m-1)(m-2)(m-3)}{n(n-1)(n-2)(n-3)}$ as a fraction with a rational denominator when $m=\sqrt{-3}+1$, $n=\sqrt{-2}+1$.
[Cal. F. A. 1886]

20. Prove the following :
(i) $(x^2+a^2)^3=(x^3-3a^2x)^2+(3ax^2-a^3)^2$.
(ii) $(x^2+a^2)^4=(x^4-6x^2a^2+a^4)^2+(4x^3a-4xa^3)^2$.
(iii) $(x^2+a^2)^5=(x^5-10x^3a^2+5xa^4)^2+(5x^4a-10x^2a^3+a^5)^2$.
(iv) $(x^2+a^2)^6=(x^6-15x^4a^2+15x^2a^4-a^6)^2$
$$+(6x^5a-20x^3a^3+6xa^5)^2.$$

21. If $x=\sqrt{[-q+p\sqrt{\{-q+p\sqrt{(-q+\cdots\cdots \textit{ad inf.}}}]}$, show that it satisfies the equation $x^2-px+q=0$.

120. The three Cube Roots of Unity.

Let $x=\sqrt[3]{1}$, then $x^3=1$.

$\therefore$ $x^3-1=0$

or $(x-1)(x^2+x+1)=0$

$\therefore$ either $x-1=0$ ... ... (1)

or, $x^2+x+1=0$ ... ... (2)

From (1), we have $x=1$;

and from (2), we have $x=\dfrac{-1\pm\sqrt{-3}}{2}$.

Thus there are three cube roots of unity, *viz.*,
$$1,\ \frac{-1+\sqrt{-3}}{2} \text{ and } \frac{-1-\sqrt{-3}}{2},$$
of which the first is real and the other two are imaginary.

Again, $\left(\frac{-1+\sqrt{-3}}{2}\right)^2=\frac{1+(-3)-2\sqrt{-3}}{4}$

$$=\frac{-2-2\sqrt{-3}}{4}$$

$$=\frac{-1-\sqrt{-3}}{2};$$

and $\left(\frac{-1-\sqrt{-3}}{2}\right)^2=\frac{1+(-3)+2\sqrt{-3}}{4}$

$$=\frac{-2+2\sqrt{-3}}{4}$$

$$=\frac{-1+\sqrt{-3}}{2}.$$

Thus we see that any one of the two imaginary cube roots of unity is the square of the other; hence, if ω denote either of the imaginary roots, the other is ω^2, and we may denote the three cube roots of unity by $1, \omega, \omega^2$.

Cor. 1. Since ω is a cube root of 1, we have $\omega^3=1$, and hence $\omega=\frac{1}{\omega^2}$; that is, *each of the two imaginary cube roots of unity is the reciprocal of the other.*

Cor. 2. Since ω is a root of the equation $x^2+x+1=0$, we have $\omega^2+\omega+1=0$,

that is, *the sum of the three cube roots of unity is zero.* This result may also be obtained by actually adding the roots; thus,

$$1+\omega+\omega^2=1+\frac{-1+\sqrt{-3}}{2}+\frac{-1-\sqrt{-3}}{2}$$

$$=1+\frac{-2}{2}=1-1=0.$$

Again, since ω^2 is a root of the equation $x^2+x+1=0$, we have $\omega^4+\omega^2+1=0$,

that is, *the sum of the squares of the three cube roots of unity is zero.* This is also evident from the fact that $\omega^4=\omega^3.\omega=\omega$, and therefore $\omega^4+\omega^2+1=\omega+\omega^2+1=0$.

121. Just like unity, every real number has three cube roots two of which are imaginary. For the cube roots of any number a^3 are those of $a^3 \times 1$, and are therefore a, $a\omega$, $a\omega^2$. The three cube roots may also be obtained directly thus :

Let $x = \sqrt[3]{a^3}$, then $x^3 = a^3$

$\therefore \quad x^3 - a^3 = 0$

or, $\quad (x-a)(x^2+ax+a^2)=0$

$\therefore$ either $\quad x-a=0$, which gives $x=a$;

or $\quad x^2+ax+a^2=0$

whence,
$$x = \frac{-a \pm \sqrt{-3a^2}}{2}$$
$$= a\left\{\frac{-1 \pm \sqrt{-3}}{2}\right\}$$
$$= a\omega \text{ and } a\omega^2.$$

Thus the three cube roots of a^3 are a, $a\omega$, $a\omega^2$.

122. Powers of ω.

Since $\omega^3 = 1$, any positive integral power of ω is equal to 1, or ω, or ω^2 ; thus

$$\omega^4 = \omega^3.\omega = 1.\omega = \omega \ ;$$
$$\omega^5 = \omega^3.\omega^2 = 1.\omega^2 = \omega^2 \ ;$$
$$\omega^6 = (\omega^3)^2 = 1^2 = 1 \ ;$$
$$\omega^{13} = (\omega^3)^4.\ \omega = 1^4.\omega = \omega \ ;$$
$$\omega^{21} = (\omega^3)^7 = 1^7 = 1 \ ; \text{ etc.}$$

In general, $\omega^n = 1$, ω or ω^2, according as n, when divided by 3, leaves 0, 1 or 2 as remainder.

Example 1. *Prove that*

(*i*) $(a-b\omega)(a-b\omega^2) = a^2+ab+b^2$.

(*ii*) $(a+b\omega)(a+b\omega^2) = a^2-ab+b^2$.

(*iii*) $(a+b\omega+c\omega^2)(a+b\omega^2+c\omega) = a^2+b^2+c^2-bc-ca-ab$.

By actual multiplication, we have

(i) $(a-b\omega)(a-b\omega^2) = a^2 - ab(\omega+\omega^2) + b^2\omega^3$
$$= a^2+ab+b^2 \ [\because \omega+\omega^2 = -1, \text{ and } \omega^3 = 1]$$

(ii) $(a+b\omega)(a+b\omega^2) = a^2+ab(\omega+\omega^2)+b^2\omega^3$
$$= a^2-ab+b^2 \ [\because \omega+\omega^2 = -1, \text{ and } \omega^3 = 1]$$

(*iii*) $(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$

$$=a^2+b^2\omega^3+c^2\omega^3+ab(\omega+\omega^2)+bc(\omega^2+\omega^4)+ca(\omega+\omega^2)$$
$$=a^2+b^2+c^2-ab-bc-ca$$
$$[\because\ \omega^3=1,\ \omega+\omega^2=-1, \text{ and } \omega^2+\omega^4=-1].$$

Conversely, these expressions may be resolved into two linear imaginary factors in the following way;

(i) $$a^2+ab+b^2=\left(a+\frac{b}{2}\right)^2+\frac{3b^2}{4}$$
$$=\left(a+\frac{b}{2}\right)^2-\left(-\frac{3}{4}b^2\right)$$
$$=\left(a+\frac{b}{2}\right)^2-\left(\frac{\sqrt{-3}}{2}b\right)^2$$
$$=\left(a+\frac{b}{2}+\frac{\sqrt{-3}}{2}b\right)\left(a+\frac{b}{2}-\frac{\sqrt{-3}}{2}b\right)$$
$$=\left\{a-\left(\frac{-1-\sqrt{-3}}{2}\right)b\right\}\left\{a-\left(\frac{-1+\sqrt{-3}}{2}\right)b\right\}$$
$$=(a-b\omega)(a-b\omega^2).$$

(ii) $$a^2-ab+b^2=\left(a-\frac{b}{2}\right)^2+\frac{3}{4}b^2$$
$$=\left(a-\frac{b}{2}\right)^2-\left(\frac{\sqrt{-3}}{2}b\right)^2$$
$$=\left(a-\frac{b}{2}+\frac{\sqrt{-3}}{2}b\right)\left(a-\frac{b}{2}-\frac{\sqrt{-3}}{2}b\right)$$
$$=\left\{a+\left(\frac{-1+\sqrt{-3}}{2}\right)b\right\}\left\{a+\left(\frac{-1-\sqrt{-3}}{2}b\right)\right\}$$
$$=(a+b\omega)(a+b\omega^2)$$

(iii) $$a^2+b^2+c^2-bc-ca-ab$$
$$=a^2-a(b+c)+(b^2-bc+c^2)$$
$$=\left(a-\frac{b+c}{2}\right)^2+(b^2-bc+c^2)-\left(\frac{b+c}{2}\right)^2$$
$$=\left(a-\frac{b+c}{2}\right)^2+3\left(\frac{b-c}{2}\right)^2$$
$$=\left(a-\frac{b+c}{2}+\sqrt{-3}.\frac{b-c}{2}\right)\left(a-\frac{b+c}{2}-\sqrt{-3}.\frac{b-c}{2}\right)$$
$$=(a+b\omega+c\omega^2)(a+b\omega^2+c\omega).$$

EXERCISE 40.

1. If ω and ω^2 are the two imaginary cube roots of unity, show that :

(i) $(1-\omega+\omega^2)(1+\omega-\omega^2)=4$.

(ii) $(1-\omega+\omega^2)^2+(1+\omega-\omega^2)^2=-4$.

(iii) $(1+\omega-\omega^2)^3=(1-\omega+\omega^2)^3=-8$.

(iv) $(a+b\omega+c\omega^2)^3+(a+b\omega^2+c\omega)^3$
$$=(2a-b-c)(2b-c-a)(2c-a-b).$$

2. Resolve $a^3+b^3+c^3-3abc$ into three linear factors.

3. If $a=x^2+2yz$, $b=y^2+2zx$ and $c=z^2+2xy$, prove that
$$a^3+b^3+c^3-3abc=(x^3+y^3+z^3-3xyz)^2.$$

4. If $X=ax+by+cz$, $Y=bx+cy+az$, $Z=cx+ay+bz$, show that $X^3+Y^3+Z^3-3XYZ$
$$=(a^3+b^3+c^3-3abc)(x^3+y^3+z^3-3xyz).$$

5. Show that the three cube roots of i are
$$-i,\ \frac{i+\sqrt{3}}{2},\ \text{and}\ \frac{i-\sqrt{3}}{2}.$$

6. Prove that $\left\{\dfrac{-1+\sqrt{-3}}{2}\right\}^n+\left\{\dfrac{-1-\sqrt{-3}}{2}\right\}^n=2$, if n be a multiple of 3, and $=-1$, if n be any other integer.

[Cal. F. A. 1892]

7. If $\alpha^2+\alpha+1=0$, show that
$$x^3-1=(x-1)(x-\alpha)(x-\alpha^2).$$ [Bom. P. E. 1893]

8. Prove that
$$m^3+n^3=(m+n)(\omega m+\omega^2 n)(\omega^2 m+\omega n)$$
$$m^3-n^3=(m-n)(\omega m-\omega^2 n)(\omega^2 m-\omega n).$$

9. Simplify : $(2+3\sqrt{-1})^4+(2-3\sqrt{-1})^4$. [Cal. Int. 1909]

10. Show that $\sqrt{[-1-\sqrt{\{-1-\sqrt{(-1-\ldots ad\ inf.)}\}}]}=\omega$ or ω^2.

11. Show that a real value of x will satisfy the equation
$$\frac{1-ix}{1+ix}=a-ib,$$
if $a^2+b^2=1$, a and b being real. [Cal. Int. 1933]

12. Show that the two roots of either of the two quadratic equations $x^2+\dfrac{\sqrt{5}+1}{2}x+1=0$ and $x^2-\dfrac{\sqrt{5}-1}{2}x+1=0$ are the squares as well as the cubes of the two roots of the other. Explain why the two equations are so related.

CHAPTER XII

PROGRESSIONS.

123. Definitions.

A number of quantities each of which is formed from one or more of the preceding quantities according to some fixed law is called a *series*; and the successive quantities forming the series are called the *terms* of the series.

Thus	2,	5,	8,	11,	14…………	(1)
	3,	6,	12,	24,	48…………	(2)
	1,	2,	3,	5,	8…………	(3)
	1,	$\frac{1}{3}$,	$\frac{1}{5}$,	$\frac{1}{7}$,	$\frac{1}{9}$…………	(4)

are all examples of series. In (1), each term is obtained by adding 3 to the preceding term; in (2), each term is twice the preceding term; in (3), each term after the second is the sum of the two preceding terms; and in (4) the reciprocal of each term is greater than the reciprocal of the preceding term by 2.

Thus we see that when the *law of formation* of successive terms is known, we can write down as many terms of the series as we please.

I. Arithmetical Progression.

124. Definition.

A number of quantities are said to be in *Arithmetical Progression* when each term is formed from the preceding term by adding to it a constant quantity. The constant quantity is called the *common difference*, and it is found by subtracting any term from the term which follows it.

The words *Arithmetical Progression* are sometimes written briefly as *A. P.*

Thus each of the following series of terms is in A. P.:

3, 7, 11, 15, … … ; common difference = 4
12, 5, -2, -9… … ; common difference = -7.

The series $a, a+b, a+2b, a+3b,$……………………
may be taken as the most general form of an A. P., the first term being a, and the common difference b.

125. The nth term.

Taking the general form of an A. P.

$$a,\ a+b,\ a+2b,\ a+3b\ldots\ldots\ldots$$

in which the first term is a, and the common difference b, we have

the 2nd term $=a+b$, or, $a+(2-1)b$,

the 3rd term $=a+2b$, or, $a+(3-1)b$,

the 4th term $=a+3b$, or, $a+(4-1)b$,

the 20th term $=a+19b$, or, $a+(20-1)b$, and so on; and generally the pth term $=a+(p-1)b$.

Hence, if n be the number of terms, and if l denote the last or nth term, we have

$$\boldsymbol{l=a+(n-1)b} \qquad \ldots \qquad \ldots \quad \text{(I)}$$

By means of this relation, when any three of the quantities a, b, n, l are given, we may find the remaining one. Thus, when a, n, and l are given, we have the common difference $b=\dfrac{l-a}{n-1}$, and so on.

Cor. When any two terms of an A. P. are given, the whole series can be determined.

For suppose the pth term and the qth term are given, and equal to h, k, respectively; then we have

$$h=a+(p-1)b,$$
$$k=a+(q-1)b,$$

which are a pair of simultaneous equations from which a and b can be found and hence the whole series determined.

Example 1. *Find the 30th term of the series* 7, 12, 17,......

Here $a=7$, $b=5$, and $n=30$.

$\therefore$ the 30th term $=7+(30-1)5=7+29\times5=152$. [Formula (I)]

Example 2. *If the 20th term of a series in A. P. whose first term is* 3, *is* 79, *find the common difference.*

Here $a=3,\ n=20,\ l=79.$

$$79=3+(20-1)b=3+19b$$

$\therefore \quad 19b=79-3=76$

$\therefore \quad b=4.$

Hence, 4 is the common difference.

Example 3. *The* 11*th and* 25*th terms of an A. P. are respectively* 23 *and* 65 ; *find the series.*

Let a be the first term, and b the common difference.
Then the 11th term $= a+(11-1)b = a+10b$;
and the 25th term $= a+(25-1)b = a+24b$.
Hence by the question, $a+10b=23$ and $a+24b=65$.
Solving these two equations in a and b, we have

$$a = -7 \text{ and } b=3$$

$\therefore$ the series is $-7, -4, -1, 2, 5, \ldots \ldots$

EXERCISE 41.

1. Find

(i) the 20th term of the series 5, 9, 13, 17,······

(ii) the 25th term of the series 16, 11, 6, 1, -4,...

(iii) the 30th term of the series ·1, ·09, ·08,······

(iv) the nth term of the series $a-b$, a, $a+b$, $a+2b$,···

(v) the nth term of the series n, $2n+1$, $3n+2$,...

2. If the 31st term of a series in A. P., whose first term is 13, is -77, find the common difference.

3. Which term of the series 10, 4, -2, -8,···is -104 ?

4. Find the common difference of an A. P. of which the first term is 1 and the tenth term is 10. [Cal. 1925]

5. If the nth term of the series 3, 10, 17,···is the same as the nth term of the series 63, 65, 67,··· ; find n.

6. The first term of an A. P. is 2, the 20th term is 59. Find the 32nd term. [Dac. 1930]

7. The first term of an A. P. is 6 and the common difference is 2. Find the 15th term. [Cal. 1922]

8. The pth term of an A. P. is q, and the qth term is p. Find the rth, $(p+q)$th and $(p+q-1)$th terms.

9. The pth term of an A. P. is $3p-2$, where p is any positive integer. Determine the series.

10. The pth term of an A. P. is m and the qth term is n. Find the rth term.

126. Arithmetic Means.

(*a*) *Definition.*

When three quantities are in A. P., the middle one is called the *Arithmetic Mean* between the other two. Thus, 4, 7, 10 are in A. P. ; hence 7 is the Arithmetic mean between 4 and 10.

When any number of quantities are in A. P. the terms intermediate between the two extreme ones are called the *Arithmetic Means* between the two extreme terms.

Thus 2, 7, 12, 17. 22, 27, 32, 37 are in A.P.; hence 7, 12, 17, 22, 27, 32 are the six Arithmetic means between 2 and 37.

The usual abbreviation for Arithmetic Means is A. M.

(*b*) *To find the Arithmetic mean of two given quantities.*

Let a and b be the two given quantities and x be the required A. M.; then a, x, b are in A. P.

$\therefore$ common difference $= x - a = b - x$

$$\therefore \quad 2x = a + b$$

$$\therefore \quad x = \frac{a+b}{2}.$$

Thus, *the A.M. between two quantities is equal to half their sum.*

(*e*) *To insert p Arithmetic means between two given quantities a* and l.

Here we require a series in A. P. beginning with a and ending with l and having p terms between a and l. Thus the total number of terms in the complete series, including the two extreme terms a and l, will be $p+2$.

Let b be the common difference.

Then since l is the $(p+2)$th term of the series of which a is the first term and b the common difference, we have

$$l = a + (p+2-1)b = a + (p+1)b,$$

whence $$b = \frac{l-a}{p+1}.$$

Therefore the required Arithmetic means are

$$a + \frac{l-a}{p+1}, \qquad a + \frac{2(l-a)}{p+1}, \qquad \dots \quad \dots \quad l - \frac{l-a}{p+1}.$$

Example 1. *Find the Arithmetic mean between $a+b$ and $a-b$.*

Let x be the required mean. Then $a+b$, x, $a-b$ are in A. P.

$$\therefore \quad x - (a+b) = (a-b) - x$$

$$\therefore \quad 2x = (a+b) + (a-b) = 2a$$

$\therefore$ $x = a$, which is the A. M. required.

Example 2. *Insert 5 Arithmetic means between 4 and 19.*

Let x_1, x_2, x_3, x_4, x_5 be the required means.

Then $4, x_1, x_2, x_3, x_4, x_5, 19$ are in A.P.; that is, 19 is the 7th term of the series in A. P. of which the 1st term is 4. Hence if b = the common difference, we must have

$$19 = 4 + (7-1)\,b = 4 + 6b$$

$$\therefore \quad b = \tfrac{15}{6} = 2\tfrac{1}{2}.$$

Hence
$$x_1 = 4 + 2\tfrac{1}{2} = 6\tfrac{1}{2},$$
$$x_2 = 4 + 2 \times 2\tfrac{1}{2} = 9,$$
$$x_3 = 4 + 3 \times 2\tfrac{1}{2} = 11\tfrac{1}{2},$$
$$x_4 = 4 + 4 \times 2\tfrac{1}{2} = 14,$$
$$x_5 = 4 + 5 \times 2\tfrac{1}{2} = 16\tfrac{1}{2}.$$

$\therefore$ the required means are $6\tfrac{1}{2}$, 9, $11\tfrac{1}{2}$, 14 and $16\tfrac{1}{2}$.

EXERCISE 42.

1. Find the Arithmetic mean between

(i) 7 and 15. (ii) 10 and -9.

(iii) $a+2b$ and $2a+b$. (iv) $(a+b)^2$ and $-(a-b)^2$.

(v) $(a+b)(x+y)$ and $(a-b)(x-y)$.

2. Insert:

(i) 4 Arithmetic means between 5 and 35.

(ii) 7 Arithmetic means between 1 and 41. [Cal. 1914]

(iii) 10 Arithmetic means between 2 and 57.

(iv) 3 Arithmetic means between a and b.

(v) 3 Arithmetic means between $a-b$ and $13a+15b$.

3. There are n Arithmetic means between 20 and 80, such that the first mean : the last mean = 1 : 3 ; find n.

127. Sum to n terms.

Let the series be a, $a+b$, $a+2b$.........to n terms, so that a is the first term and b the common difference ; and let s denote the required sum.

Then $s = a + (a+b) + (a+2b) + \ldots\ldots + \{a + (n-1)b\}$ ... (1)

Again, writing the series in the reverse order, we have

$$s = l + (l-b) + (l-2b) + \cdots\cdots + \{l - (n-1)b\} \quad \cdots \quad (2)$$

where $\quad l$ = the nth term of the series $= a + (n-1)b$.

adding (1) and (2), we have

$$2s=(a+l)+(a+l)+(a+l)+\ldots\ldots\text{to } n \text{ terms}=n(a+l).$$

$$s=\frac{n}{2}(a+l) \qquad \ldots \qquad \ldots \quad \text{(II)}$$

Also since $l=a+(n-1)b$, we have from (II),

$$s=\frac{n}{2}\{a+a+(n-1)b\}$$

$$=\frac{n}{2}\{2a+(n-1)b\} \qquad \cdots \quad \text{(III)}$$

Formula (II) should be used when the first and last terms are given; and formula (III) should be used when the first term and the common difference are given.

Example 1. *Find the sum of* 50 *terms of the series*

1, 2, 3, 4.······

The first term of the series $=1$; the last term $=50$.

Hence, the required sum $=\frac{50}{2}(1+50)=25\times 51=1275$.

Example 2. *Find the sum of* 30 *terms of the series* 4, 7, 10,···

The first term of the series $=4$; the common difference $=7-4=3$.

∴ the sum $=\frac{30}{2}\{2\times 4+(30-1)\times 3\}=15(8+87)=15\times 95=1425$.

EXERCISE 43.

Find the sum of :

1. $1+2+3+4+$ to 100 terms.
2. $2+4+6+$ to 50 terms.
3. $1+3+5+7+$ to 30 terms.
4. $35+32+29+$ to 24 terms.
5. $4\frac{1}{4}+5\frac{3}{4}+7\frac{1}{4}+$ to 16 terms.
6. $2+3\frac{1}{3}+4\frac{2}{3}+$ to 60 terms.
7. $5{\cdot}1+4{\cdot}7+4{\cdot}3+$ to 30 terms.
8. $3{\cdot}5+2{\cdot}5+1{\cdot}5+$ to 14 terms.
9. $3+4+8+9+13+14+18+19+\cdot$ to 20 terms. [Cal. F.A. 1881]
10. $\sqrt{2}+\sqrt{2}(1-\sqrt{2})+\sqrt{2}(1-2\sqrt{2})+\cdots$ to 21 terms.
11. $\dfrac{1}{\sqrt{2}+1}+\sqrt{2}+\dfrac{1}{\sqrt{2}-1}+\cdots$ to n terms.
12. $n+(n-1)+(n-2)+\cdots\cdots$ to n terms.
13. $(a-2b)+(2a-3b)+(3a-4b)+\cdots$ to n terms.
14. $(x+y)^2+(x^2+y^2)+(x-y)^2+\cdots$ to n terms. [Cal. F.A. 1880]

15. Find the sum of all integral numbers from m to n.

16. Find the sum of all even numbers from 101 to 999.

17. Find the sum of all odd numbers of four digits which are divisible by 9.

18. Sum 15 terms of an A. P. of which 20 is the middle term.

19. Find the sum of the first n terms of the series whose rth term is $3r-1$.

20. Sum to 53 terms the series whose $(n+1)$th term is $3n+2$.

21. If 4 be added to the sum of any number of terms of the series 21, 39, 57, 75,... show that the result will be a square number.

22. If 1 be added to the sum of any number of terms of the series 8, 16, 24, 32,... show that result will be a square number.

23. In an A. P. consisting of $2n+1$ terms, show that the sum of the odd terms is to the sum of the even terms as $n+1 : n$.

24. If the 12th term of an A. P. is 5 and the 50th term is $9\frac{3}{4}$, show that the sum of n terms of the series is $\frac{n}{16}(n+57)$.

25. Show that the sum of any $2n+1$ consecutive integers is divisible by $2n+1$.

128. The Natural Numbers.

The numbers 1, 2, 3, 4$\cdots$are called the *natural numbers*.

The following series relating to natural numbers are important, and the results should be remembered.

Example 1. *Find the sum of the first n natural numbers.*

Let s denote the sum.

Then $s=1+2+3+\cdots\cdots\cdots\cdots+n=\frac{n}{2}(1+n)=\frac{n(n+1)}{2}$.

Example 2. *Find the sum of the first n odd natural numbers.*

Let s denote the sum; then

$$s=1+3+5+7+\cdots\cdots\text{to } n \text{ terms}=\frac{n}{2}\left\{2+(n-1)\times 2\right\}$$

$$=\frac{n}{2}\times 2n$$

Example 3. *Find the sum of the squares of the first n natural numbers.* [*Cal.* 1915]

Let s denote the sum : then $s=1^2+2^2+3^2+4^2+\cdots\cdots+n^2$.

We have $n^3-(n-1)^3=3n^2-3n+1$, identically.

Hence, putting 1, 2, 3,$\cdots$for n, we have

$$
\begin{aligned}
1^3-0^3&=3.1^2-3.1+1,\\
2^3-1^3&=3.2^2-3.2+1,\\
3^3-2^3&=3.3^2-3.3+1,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
(n-1)^3-(n-2)^3&=3.(n-1)^2-3.(n-1)+1,\\
n^3-(n-1)^3&=3.n^2-3.n+1.
\end{aligned}
$$

Adding together the vertical columns we have

$$
\begin{aligned}
n^3-0^3&=3[1^2+2^2+3^2+\cdots\cdots+(n-1)^2+n^2]\\
&\qquad-3[1+2+3+\cdots\cdots+(n-1)+n]+n\\
&=3s-3\times\frac{n(n+1)}{2}+n \qquad \text{[See Example 1.]}
\end{aligned}
$$

$$
\begin{aligned}
3s&=n^3-n+\frac{3n(n+1)}{2}=n(n+1)(n-1+\tfrac{3}{2})\\
&=\frac{n(n+1)(2n+1)}{2}.
\end{aligned}
$$

Hence $$s=\frac{n(n+1)(2n+1)}{6}.$$

Example 4. *Find the sum of the cubes of the first n natural numbers.* [*Cal. F. A.* 1904.]

Let s denote the sum ; then $s=1^3+2^3+3^3+4^3+\cdots\cdots+n^3$.

We have $n^4-(n-1)^4=4n^3-6n^2+4n-1$, identically.

Hence, putting 1, 2, 3, $\cdots\cdots$for n, we have

$$
\begin{aligned}
1^4-0^4&=4.1^3-6.1^2+4.1-1,\\
2^4-1^4&=4.2^3-6.2^2+4.2-1,\\
3^4-2^4&=4.3^3-6.3^2+4.3-1,\\
&\\
(n-1)^4-(n-2)^4&=4.(n-1)^3-6.(n-1)^2+4.(n-1)-1,\\
n^4-(n-1)^4&=4.n^3-6.n^2+4.n-1.
\end{aligned}
$$

Adding together the vertical columns, we have

$$
\begin{aligned}
n^4&=4(1^3+2^3+3^3+\cdots\cdots+n^3)-6(1^2+2^2+3^2+\cdots+n^2)\\
&\qquad+4(1+2+3+\cdots\cdots+n)-n
\end{aligned}
$$

$$=4s-6.\frac{n(n+1)(2n+1)}{6}+4.\frac{n(n+1)}{2}-n.$$

[See Examples 1 and 3.]

$$\therefore \quad 4s=n^4+n+n(n+1)(2n+1)-2n(n+1)$$
$$=n(n+1)\{(n^2-n+1)+(2n+1)-2\}$$
$$=n(n+1)(n^2+n)$$
$$=n(n+1)n(n+1)$$
$$=n^2(n+1)^2$$

$$\therefore \quad s=\frac{n^2(n+1)^2}{4}=\left\{\frac{n(n+1)}{2}\right\}^2.$$

Thus we see that

$$1^3+2^3+3^3+\cdots\cdots+n^3=\left\{\frac{n(n+1)}{2}\right\}^2=(1+2+3+\cdots+n)^2$$

that is, *the sum of the cubes of the first n natural numbers is equal to the square of the sum of the first n natural numbers.*

129. The Σ Notation.

A very convenient and brief notation is sometimes made use of to indicate the sum of a series whose terms are of a given type. The Greek letter Σ (pronounced *sigma*) is used, and put before the typical term. Thus Σn stands for the sum of the series whose typical (or nth) term is n, so that putting $n=1, 2, 3, \cdots$ we see that

$$\Sigma n=1+2+3+\cdots\cdots+n.$$

Similarly, $\Sigma n^2=1^2+2^2+3^2+\cdots\cdots+n^2$;

$$\Sigma n^3=1^3+2^3+3^3+\cdots\cdots+n^3;$$

$$\Sigma n(n+1)=1.2+2.3+3.4+\cdots\cdots+n(n+1);$$

and so on.

Example 1. *Sum the series* $1.2.3+2.3.4+3.4.5+\cdots\cdots\cdots$ *to n terms.* (*Cal. F. A.* 1890)

The nth term of the series evidently

$$=n(n+1)(n+2)=n(n^2+3n+2)=n^3+3n^2+2n.$$

Hence, $s=\Sigma n^3+3\Sigma n^2+2\Sigma n$

$$=\frac{n^2(n+1)^2}{4}+3.\frac{n(n+1)(2n+1)}{6}+2.\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)}{4}\left\{n(n+1)+2(2n+1)+4\right\}$$

$$=\frac{n(n+1)}{4}(n^2+5n+6)$$

$$=\frac{n(n+1)(n+2)(n+3)}{4}.$$

EXERCISE 44.

Sum the following series to n terms :

1. $1^2+4^2+7^2+10^2+\cdots\cdots$
2. $2^2+4^2+6^2+8^2+\cdots\cdots$
3. $5^2+8^2+11^2+14^2+\cdots$
4. $3^2+5^2+7^2+9^2+\cdots\cdots$
5. $5^2+9^2+13^2+17^2+\cdots$
6. $2^3+5^3+8^3+11^3+\cdots$
7. $1^3+3^3+5^3+7^3+\cdots\cdots$
8. $8^2+5^2+2^2+1^2+4^2+\cdots$
9. $1.2^2+2.3^2+3.4^2+\cdots\cdots$
10. $1.3+3.5+5.7+7.9+\cdots\cdots$
11. $1+(1+2)+(1+2+3)+\cdots$
12. $1.2+2.3+3.4+\cdots\cdots$
13. $1.3.5+3.5.7+5.7.9+\cdots$
14. $1^4+2^4+3^4+\cdots\cdots$
15. $4+7+12+19+28+\cdots\cdots$
16. $3+7+13+21+31+\cdots\cdots$
17. $1^5+2^5+3^5+\cdots$
18. $1^6+2^6+3^6+\cdots$
19. Sum the series $\dfrac{1}{1.2}+\dfrac{1}{2.3}+\dfrac{1}{3.4}+\cdots$ to n terms.
20. Sum the series $\dfrac{1}{2.5}+\dfrac{1}{5.8}+\dfrac{1}{8.11}+\dfrac{1}{11.14}+\cdots\cdots$ to n terms.

II. Geometrical Progression.

130. Definition.

A number of quantities are said to be in *Geometrical Progression*, when each quantity bears to the preceding one a constant ratio. This constant ratio is called the *common ratio*, and it is found by dividing any term by the term which precedes it.

The words *Geometrical Progression* ars sometimes written briefly as *G. P.*

Thus the following series of terms are in G. P. :

$1, \quad 3, \quad 9, \quad 27, \quad 81, \quad \cdots$; common ratio $=3$;

$1, \quad \frac{1}{2}, \quad \frac{1}{4}, \quad \frac{1}{8}, \quad \frac{1}{16}, \quad \cdots$; common ratio $=\frac{1}{2}$;

$2, \quad -1, \quad \frac{1}{2}, \quad -\frac{1}{4}, \quad \frac{1}{8}, \quad \cdots$; common ratio $=-\frac{1}{2}$.

The series $\qquad a, ar, ar^2, ar^3, \cdots\cdots$

may be taken as the most general form of a G. P., the first term being a and the common ratio r.

It is evident that *if any number of quantities are in G. P. they are in continued proportion.*

131. The *n*th term.

Let us take the general form of a series in G. P.

$$a, ar, ar^2, ar^3.$$

in which the first term is a, and the common ratio r.

We have the 2nd term $= ar$,
 „ „ 3rd „ $= ar^2$,
 „ „ 4th „ $= ar^3$,
and generally, pth „ $= ar^{p-1}$.

Hence if n be the number of terms, and if l denote the last or nth term, we have

$$l = ar^{n-1} \quad \ldots \quad \ldots \quad \text{(I)}$$

By means of this formula, when any three of the quantities a, r, n, l are given, we may find the remaining one.

Cor. 1. The product of any two terms of a G. P. equidistant from the beginning and the end is always the same, and equal to the product of the first and the last term.

For the pth term from the beginning $= ar^{p-1}$.

Similarly, „ „ end $= \dfrac{l}{r^{p-1}}$.

Hence, their product $= al$ (*i.e.*, the product of the first term and the last term) = a constant.

Cor. 2. When any two terms of a G. P. are given, the whole series can be determined.

For, let the pth term be h and the qth term k, then we have from formula (I),

$$h = ar^{p-1} \text{ and } k = ar^{q-1}.$$

Dividing, $\dfrac{h}{k} = r^{p-q}$ $\qquad \therefore\ r = \left(\dfrac{h}{k}\right)^{\frac{1}{p-q}}$.

The value of r being known, a can be obtained; and a and r being known the whole series is determined.

Example 1. *Find the* 16*th term of the series* 3, 6, 12, 24....

Here, $a = 3$, $r = 2$ and $n = 16$.

$\therefore$ the 16th term $= 3 \times 2^{15} = 3 \times 32768 = 98304$. [Formula (I)]

Example 2. *The 7th and 10th terms of a series in G. P. are respectively -27 and 729. Find the first term and the common ratio.*

Let a be the first term, and r the common ratio.

Then, by the question, $\quad -27 = ar^6 \quad \ldots \quad (1)$

$$729 = ar^9 \quad \ldots \quad (2)$$

Dividing (2) by (1), $\quad r^3 = -27$

$\therefore \quad r = -3.$

Substituting the value of r in (1), we have

$$-27 = a(-3)^6 = 729a$$

$\therefore \quad a = -\frac{27}{729} = -\frac{1}{27}.$

Hence the first term $= -\frac{1}{27}$, and common ratio $= -3$.

EXERCISE 45.

1. Find (i) the 12th term of the series $2, 4, 8, \cdots$
   (ii) the 8th term of the series $4\frac{1}{5}, 8\frac{2}{5}, 16\frac{4}{5}, \cdots$
   (iii) the 10th term of the series $2, -1, \frac{1}{2}, -\frac{1}{4}, \cdots$
   (iv) the 12th term of the series $2, -2\sqrt{3}, 6, \cdots$
   (v) the 20th term of the series $\sqrt{3}, \sqrt{6}, 2\sqrt{3}, \cdots$

2. Find the nth term of series : $\sqrt{3} + \frac{1}{\sqrt{3}} + \frac{1}{3\sqrt{3}} + \cdots$ [Cal. 1886]

3. Find the nth term of the series $2\frac{3}{4}, 1\frac{5}{6}, 1\frac{2}{9}, \cdots$

4. The 4th and 9th terms of a series in G. P. are $\frac{1}{2}$ and $\frac{16}{243}$ respectively. Find the first term, the common ratio and the fifth term of the series.

5. Which term of the series $\frac{1}{2}, -\frac{1}{3}, \frac{2}{9} \cdots$ is $-\frac{64}{2187}$?

6. Find the 9th term of a series in G. P. whose 4th term is 1 and 7th term is $\frac{1}{8}$.

7. Which term of the series $\frac{8}{9}, \frac{4}{3}\sqrt{\frac{2}{3}}, \frac{4}{3}, \cdots$ is $\sqrt{6}$?

8. The pth term of a G. P. is a, and the qth term is b. Find the nth term.

9. In a G. P. if the $(p+q)$th term $= m$, and the $(p-q)$th term $= n$, find the pth and qth terms. [Bom. P. E. 1888]

10. Determine the G.P. whose rth term is

$$(-1)^{r-1} 2^{2r} 3^{1-r}.$$

11. In a G. P. show that the first term is to the nth term as the $(n-1)$th power of the first is to the $(n-1)$th power of the second.

12. From three given numbers which are in G. P., three other numbers in G. P. are subtracted and the remainders are found to be also in G. P. Prove that the three series have the same common ratio. [Bom. P. E. 1890]

13. A series of terms are in G. P., the sum of the first two is $1\frac{1}{3}$, and the sum of the next two is 12. Find the series.

14. The fifth term of a G. P. is 4 times the 3rd, and the sum of the first two terms is -4. Find the series.

15. If there are 6 terms in a G. P, prove that the product of the first and last is equal to the product of the third and fourth. [Pun. I. E. 1893]

132. Geometric Means.

(*a*) *Definition.*

When three quantities are in G. P., the middle one is called the *Geometric Mean* of the other two. Thus, 3, 6, 12 are in G. P.; hence, 6 is the Geometric mean of 3 and 12.

When any number of quantities are in G. P., the terms intermediate between the two extreme ones are called the *Geometric Means* between the two extreme terms. Thus, 2, 6, 18, 54, 162, 486, 1458 are in G. P.; and hence 6, 18, 54, 162, 486 are the five Geometric means between 2 and 1458.

The usual abbreviation for Geometric Means is G. M.

(*b*) *To find the Geometric mean of two given quantities.*

Let a and b be the given quantities and x be the required G. M.; then a, x, b are in G. P., and therefore by definition of G. P., we have

$$\frac{x}{a}=\frac{b}{x}, \text{ or, } x^2=ab. \quad \therefore \quad x=\pm\sqrt{ab}.$$

Thus, *the Geometric mean of any two quantities is the square root of their product.*

In other words, *the Geometric mean of two quantities is the same as their mean proportional.*

NOTE. *The Arithmetic mean of two unequal positive quantities is greater than their Geometric mean.*

For, if a and b be the two quantities, $\frac{a+b}{2}$ is their A. M. and $\sqrt{ab}$ is their G. M., and $\frac{a+b}{2}-\sqrt{ab}=\frac{1}{2}(a+b-2\sqrt{ab})=\frac{1}{2}(\sqrt{a}-\sqrt{b})^2$, which is always positive, if a and b are unequal and positive.

Hence $$\frac{a+b}{2} > \sqrt{ab}.$$

(c) *To insert p Geometric means between two given quantities a and l*

Here we require a series in G. P. beginning with a and ending with l and having p terms between a and l. Thus, the total number of terms in the complete series including the two extreme terms a and l will be $p+2$.

Let r be the common ratio.

Then since l is the $(p+2)$th term of the series of which a is the first term and r the common ratio, we must have

$$l=ar^{p+1} \qquad \text{[Formula (I), Art. 130]}$$

$$r^{p+1}=\frac{l}{a}, \quad i.e., \; r=\left(\frac{l}{a}\right)^{\frac{1}{p+1}}.$$

The means may now be easily determined; for they are ar, ar^2, ar^3, ... ar^{p-1}, ar^p. Hence, the required means are

$$a\left(\frac{l}{a}\right)^{\frac{1}{p+1}},\ a\left(\frac{l}{a}\right)^{\frac{2}{p+1}},\ a\left(\frac{l}{a}\right)^{\frac{3}{p+1}},\ \ldots a\left(\frac{l}{a}\right)^{\frac{p-1}{p+1}},\ a\left(\frac{l}{a}\right)^{\frac{p}{p+1}}.$$

Example 1. *Find the Geometric mean of 2 and 6.*

The required Geometric mean $=\pm\sqrt{2\times 6}=\pm 2\sqrt{3}$.

Example 2. *Insert 5 Geometric means between 2 and 54.*

Let x_1, x_2, x_3, x_4, x_5 be the required means.

Then $2, x_1, x_2, x_3, x_4, x_5, 54$ are in G. P. so that 54 is the 7th term of a series in G. P., of which the 1st term is 2.

Hence if r is the common ratio, we have

$$54=2\times r^6. \quad \therefore r^6=27, \text{ whence } r=\pm\sqrt{3}.$$

Hence
$$x_1=2\times(\pm\sqrt{3})=\pm 2\sqrt{3}, \quad x_2=2\times(\pm\sqrt{3})^2=6$$
$$x_3=2\times(\pm\sqrt{3})^3=\pm 6\sqrt{3}, \quad x_4=2\times(\pm\sqrt{3})^4=18$$
$$x_5=2\times(\pm\sqrt{3})^5=\pm 18\sqrt{3}.$$

That is, when the common ratio is $\sqrt{3}$, the means are $2\sqrt{3}$, 6, $6\sqrt{3}$, 18, $18\sqrt{3}$; and when the common ratio is $-\sqrt{3}$, the means are $-2\sqrt{3}$, 6, $-6\sqrt{3}$, 18, $-18\sqrt{3}$.

EXERCISE 46.

1. Find the Geometric mean between

(i) 5 and 15. (ii) 3 and 12.

(iii) $2\sqrt{3}$ and $6\sqrt{3}$. (iv) ax and bx.

(v) $x^2 - xy$ and $xy - y^2$. (vi) $(2x-5)^2$ and $(3x+2)^2$.

2. Insert

(i) 3 Geometric means between 4 and 324. [Cal. F. A. 1890]

(ii) 4 Geometric means between $2\frac{1}{2}$ and $\frac{80}{243}$.

(iii) 4 Geometric means between $-\frac{1}{12}$ and $\frac{8}{3}$.

(iv) 5 Geometric means between $\frac{16}{5}$ and $\frac{1}{20}$.

(v) 6 Geometric means between $\frac{64}{81}$ and $13\frac{1}{2}$.

(vi) 4 Geometric means between $1\frac{4}{5}$ and $\frac{32}{135}$.

(vii) 7 Geometric means between 8 and $\frac{1}{2}$.

(viii) 3 Geometric means between 25 and 164025. [Pat. 1919]

3. Show that the product of the n Geometric means inserted between the quantities a and c is $(ac)^{\frac{n}{2}}$.

4. Show that, if between every two consecutive terms of a G. P. a fixed number of G. M.'s be inserted, the whole will form a G. P.

5. Insert between 6 and 16 two numbers such that the first three may be in A. P. and the last three in G. P.

6. If $a : b = 2+\sqrt{3} : 2-\sqrt{3}$, the Arithmetic mean between a and b will be twice the Geometric mean. [All. I. E. 1895]

7. Show that the $2n$th term of any Geometrical series is the mean proportional between the nth and $3n$th terms. [Cal. F. A. 1877]

8. The Arithmetic mean between two numbers is 15 and their Geometric mean is 9. Find the numbers. [Cal. 1926]

133. Sum to *n* terms.

Let the series be $a, ar, ar^2, ar^3, \cdots\cdots$ to n terms, so that a is the first term and r the common ratio ; and let s denote the required sum. Then

$$s = a + ar + ar^2 + ar^3 + \cdots\cdots + ar^{n-1}$$

$$\therefore \quad sr = \quad ar + ar^2 + ar^3 + \cdots\cdots + ar^{n-1} + ar^n.$$

Hence, by subtraction,

$$s - sr = a - ar^n$$

$$\therefore \quad s(1-r) = a(1-r^n)$$

$$\left.\begin{aligned} s &= \frac{a(1-r^n)}{1-r}, \text{ when } r<1 \\ \text{or,} \quad &= \frac{a(r^n-1)}{r-1}, \text{ when } r>1 \end{aligned}\right\} \qquad \text{(II)}$$

If l be the last or nth term of the series, we have

$$l = ar^{n-1}$$

$$s = \frac{a-lr}{1-r} \text{ or } \frac{lr-a}{r-1} \quad \cdots \quad \text{(III)}$$

Example 1. *Find the sum of* $-\frac{1}{12}+\frac{1}{6}-\frac{1}{3}+\ldots$ *to* 6 *terms.*

The common ratio $= \frac{1}{6} \div (-\frac{1}{12}) = -2$. Hence the reqd. sum

$$= \frac{-\frac{1}{12}\{1-(-2)^6\}}{1-(-2)} = \frac{-\frac{1}{12}(1-64)}{3} = \frac{63}{12\times 3} = \frac{7}{4} = 1\tfrac{3}{4}.$$

Example 2. *Find the sum of* $8+4\sqrt{2}+4+\cdots$ *to* 10 *terms.*

The common ratio $= 4\sqrt{2} \div 8 = \dfrac{1}{\sqrt{2}}$.

$$\therefore \text{ the required sum} = \frac{8\left\{1-\left(\frac{1}{\sqrt{2}}\right)^{10}\right\}}{1-\frac{1}{\sqrt{2}}} = \frac{8\left(1-\frac{1}{32}\right)}{\frac{\sqrt{2}-1}{\sqrt{2}}} = \frac{\frac{31}{4}\sqrt{2}}{\sqrt{2}-1}$$

$$= \frac{31}{4}\sqrt{2}(\sqrt{2}+1) = \frac{31}{2} + \frac{31}{4}\sqrt{2}.$$

EXERCISE 47.

Find the sum of :

1. $2-4+8-\cdots$ to 8 terms.
2. $1+3+9+27+\cdots$ to 12 terms.
3. $3-1+\frac{1}{3}-\cdots$ to 6 terms.
4. $1-\frac{1}{2}+\frac{1}{4}-\frac{1}{8}+\cdots$ to 12 terms.
5. $1-2+2^2-2^3+\cdots$ to 10 terms.
6. $-2+2\frac{1}{2}-3\frac{1}{8}+\cdots$ to 6 terms.
7. $1\frac{1}{2}+2\frac{1}{4}+3\frac{3}{8}+\cdots$ to 8 terms.
8. $\frac{1}{\sqrt{2}}-1+\sqrt{2}-\cdots$ to 10 terms.
9. $\sqrt{2}+\sqrt{6}+\sqrt{18}+\cdots$ to 12 terms.
10. $\frac{1}{\sqrt{3}}+1+\frac{3}{\sqrt{3}}+\cdots$ to 18 terms. [Pun. I. E. 1891]

Sum the following series to n terms :

11. $2-1+\frac{1}{2}-\frac{1}{4}+\cdots$

12. $2+\sqrt{2}+1+\frac{1}{\sqrt{2}}+\cdots$ [Pun. I. E. 1890]

13. $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\cdots$

14. $\cdot 3+\cdot 03+\cdot 003+\cdot 0003+\cdots$

15. $\frac{\sqrt{3}+1}{\sqrt{3}-1}+1+\frac{\sqrt{3}-1}{\sqrt{3}+1}+\cdots$

16. $\frac{2}{3}-\sqrt{\frac{2}{3}}+1\cdots$ [Cal. F. A. 1865]

17. $\frac{a-b}{a+b}+1+\frac{a+b}{a-b}+\cdots$

18. $(a^2-b^2)+(a+b)+\frac{a+b}{a-b}+\cdots$

19. $a^p+a^{p+q}+a^{p+2q}+\cdots$

20. $a-ar+ar^2-ar^3+\cdots$ [Cal. F. A. 1865]

21. $(\sqrt{2}+1)+1+(\sqrt{2}-1)+(3-2\sqrt{2})+\cdots$

22. Find the sum of $\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{9}+\frac{1}{8}-\frac{1}{27}+\cdots$ to $2n$ terms.

23. Sum to n terms (i) the series whose rth term is $2.\ 3^r$; and (ii) the series whose mth term is $(-1)^m a^{4m}$.

24. Find the sum of the n Geometric means inserted between a and c.

25. Find the sum of the reciprocals of n terms of a G. P. of which the first term is a and common ratio r.

134. Sum to Infinity

Let the series be $a+ar+ar^2+ar^3+\cdots$, and let s denote the sum of an infinite number of terms of this series.

(i) If $r>1$, it is obvious that the sum required is infinitely large ; that is, $s=\infty$.

(ii) If $r=1$, we have

$$s=a+a+a+\cdots\text{to infinity}=a\times\infty=\infty. \quad \text{[Art. 11]}$$

(iii) If $r<1$, *i.e.*, if r be a proper fraction, we have the sum to n terms

$$=\frac{a(1-r^n)}{1-r}=\frac{a}{1-r}-\frac{ar^n}{1-r}. \quad \text{[Art. 133]}$$

Now, as n increases, the fraction r^n continually diminishes since $r<1$; hence, by sufficiently increasing the value of n, we can make r^n and therefore $\frac{ar^n}{1-r}$ as small as we please ; and therefore ultimately when n is infinitely large, the sum of

n terms of the series differs from $\frac{a}{1-r}$ by a quantity which is less than any assigned quantity however small. Hence, the sum of an infinite number of terms of the G. P.

$$=\frac{a}{1-r} \qquad \cdots \qquad \cdots \qquad \text{(IV)}$$

Thus, in a series in G. P., if the common ratio is greater than or equal to unity, the sum to infinity $=\infty$; and if the common ratio is less than unity, the sum to infinity $=\frac{a}{1-r}$, a being the first term and r the common ratio.

NOTE. *Convergent and Divergent Series.*

An infinite series whose sum is infinite is called a ***divergent*** series, and an infinite series whose sum is a finite quantity is called a ***convergent*** series. Thus *an infinite Geometric series is convergent only if the common ratio is less than unity.* (See Chap. XIII. Sec. III.)

Example 1. *Sum to infinity* : $1-\frac{2}{3}+\frac{4}{9}-\frac{8}{27}+\cdots$

Here the first term $=1$; common ratio $=-\frac{2}{3}$.

Hence the required sum $=\frac{1}{1+\frac{2}{3}}=\frac{1}{\frac{5}{3}}=\frac{3}{5}$.

Example 2. *Sum to infinity* : $\sqrt{3}+\frac{1}{\sqrt{3}}+\frac{1}{3\sqrt{3}}+\cdots$

[*Cal. F. A.* 1886]

The first term $=\sqrt{3}$; common ratio $=\frac{1}{\sqrt{3}}\div\sqrt{3}=\frac{1}{3}$.

Hence the sum required $=\frac{\sqrt{3}}{1-\frac{1}{3}}=\frac{\sqrt{3}}{\frac{2}{3}}=\frac{3\sqrt{3}}{2}$.

Example 3. *Sum to infinity* : $\frac{a}{1+x}+\frac{ax}{(1+x)^2}+\frac{ax^2}{(1+x)^3}+\cdots$

The first term $=\frac{a}{1+x}$;

and common ratio $=\frac{ax}{(1+x)^2}\div\frac{a}{1+x}=\frac{x}{1+x}$.

Hence the sum required $=\dfrac{\frac{a}{1+x}}{1-\frac{x}{1+x}}=\frac{a}{(1+x)-x}=a.$

135. Recurring Decimals as infinite G. P.

Recurring decimals in Arithmetic are good examples of infinite Geometric series. For example,

$$\cdot4\dot{7} = \cdot4777777\cdots \text{continued to infinity}$$
$$= \cdot4 + \cdot07 + \cdot007 + \cdot0007 + \cdots \text{to infinity}$$
$$= \frac{4}{10} + \frac{7}{100} + \frac{7}{1000} + \frac{7}{10000} + \cdots\cdots$$
$$= \frac{4}{10} + \frac{7}{100}\left(1 + \frac{1}{10} + \frac{1}{10^2} + \cdots\cdots\right).$$

Now the series within brackets $1 + \frac{1}{10} + \frac{1}{10^2} + \cdots$ forms an infinite G. P., whose first term is 1 and common ratio $\frac{1}{10}$.

$$\therefore \quad 1 + \frac{1}{10} + \frac{1}{10^2} + \cdots \text{to infinity} = \frac{1}{1 - \frac{1}{10}} = \frac{10}{9}$$

Hence $\quad \cdot4\dot{7} = \frac{4}{10} + \frac{7}{100} \times \frac{10}{9} = \frac{4}{10} + \frac{7}{90} = \frac{43}{90}.$

The value of a recurring decimal may be found practically thus :

Let $\quad s = \cdot4\dot{7} = \cdot477777\cdots\cdots$

$\therefore \quad 10s = 4\cdot77777 \cdot \cdots$

and $\quad 100s = 47\cdot77777\cdots\cdots$

$\therefore \quad 100s - 10s = 47 - 4$ [by subtraction]

or, $\quad 90s = 47 - 4 \quad \cdots$

$\therefore \quad s = \frac{47-4}{90} = \frac{43}{90}.$

Here we have the explanation of the method that is employed in Arithmetic for the conversion of a recurring decimal to a vulgar fraction.

EXERCISE 48.

Sum to infinity the following series :

1. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$
2. $\frac{1}{2} + \frac{1}{3} + \frac{2}{9} + \cdots$
3. $3 + 1\frac{1}{2} + \frac{3}{4} + \cdots$
4. $\frac{3}{5} + \frac{2}{5} + \frac{4}{15} + \cdots$
5. $\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots$
6. $9 - 6 + 4 - \cdots$
7. $\frac{3}{7} + \frac{12}{7^2} + \frac{48}{7^3} + \cdots$
8. $\frac{2}{5} + \frac{3}{5^2} + \frac{2}{5^3} + \frac{3}{5^4} + \cdots$
9. $3 - 1 + \frac{1}{3} - \frac{1}{9} + \cdots$
10. $\sqrt{2} + \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{2}} + \cdots$

[Cal. F. A. 1876]

11. $432+324+243+\dots$ [Cal. F. A. 1894]

12. $48-36+27-20\frac{1}{4}+\cdots$ [Cal. F. A. 1879]

13. $(\sqrt{2}+1)+1+(\sqrt{2}-1)+\cdots$ [Cal. F. A. 1887]

14. $(2+\sqrt{3})+1+(2-\sqrt{3})+\cdots$ [Cal. F. A. 1891]

15. $1+2x+4x^2+8x^3+\cdots(x<\frac{1}{2})$.

16. $1-x+x^2-x^3+\cdots(x<1)$.

17. Find the value of :
(i) $\cdot 2\dot{3}\dot{4}$. (ii) $\cdot 34\dot{7}$. (iii) $\cdot\dot{2}\dot{7}$. (iv) $\cdot 2\dot{0}\dot{5}$.

18. In an infinite G. P. (common ratio <1), show that each term bears a constant ratio to the sum of all the terms that follow it.

III. Harmonical Progression.

136. Definition.

A series of quantities are in Harmonical Progression when their reciprocals are in Arithmetical Progression.

The usual abbreviation for *Harmonical Progression* is *H. P.*

Thus $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, ... are in H. P.,
because $1, 2, 3, 4$. .. ... are in A. P.

and $\frac{1}{a}, \frac{1}{a+b}, \frac{1}{a+2b}$ .. ... are in H. P.,
because $a, a+b, a+2b$ .. ... are in A. P.

Since the reciprocals of the terms in H. P. are in A. P., examples in H. P. are usually solved by taking the reciprocals of the terms and using the properties of the corresponding A. P.

Cor. If a, b, c, are in H. P., then $a : c = a-b : b-c$.

For, since a, b, c are in H. P., $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$, are in A. P.

$$\therefore \frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}; \text{ or, } \frac{a-b}{ab}=\frac{b-c}{bc}; \text{ or, } \frac{a-b}{a}=\frac{b-c}{c}$$

$\therefore$ $a : c = a-b : b-c$ [by *Alternando*]

NOTE. The origin of the name *Harmonical Progression* is this: If a number of strings of lengths proportional to $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ be stretched tight and sounded together, there will arise a harmonious sound. Consequently, the word *Harmonic* is used to designate this and similar series.

137. The *n*th term.

Let the series be $\frac{1}{a}$, $\frac{1}{a+b}$, $\frac{1}{a+2b}$, ...

Then, since the *n*th term of the series a, $a+b$, $a+2b$,...... which are in A. P., is $a+(n-1)b$, the *n*th term of the series in H. P. is evidently $=\frac{1}{a+(n-1)b}$.

As in the case of an A. P., *if any two terms of an H. P. are given, the whole series can be determined.*

138. Harmonic Means.

(*a*) *Definitions.*

When three quantities are in H. P., the middle one is called the *Harmonic Mean* of the other two.

When any number of quantities are in H. P., the terms intermediate between the two extreme ones are called the *Harmonic Means* between the two extreme terms.

The usual abbreviation for Harmonic Means is H. M.

Thus $\frac{1}{3}$, $\frac{1}{5}$, $\frac{1}{7}$ are in H.P.. and so $\frac{1}{5}$ is the H. M. of $\frac{1}{3}$ and $\frac{1}{7}$; again, $\frac{1}{2}$, $\frac{1}{5}$, $\frac{1}{8}$, $\frac{1}{11}$, $\frac{1}{14}$ are in H. P., and so $\frac{1}{5}$, $\frac{1}{8}$, $\frac{1}{11}$ are the three H. M.'s between $\frac{1}{2}$ and $\frac{1}{14}$.

(*b*) *To find the Harmonic mean of any two quantities a and b.*

Let x be the H. M. required; then a, x, b are in H. P.

$$\therefore \qquad \frac{1}{a}, \frac{1}{x}, \frac{1}{b} \text{ are in A. P.,}$$

$$\therefore \qquad \frac{1}{x}-\frac{1}{a}=\frac{1}{b}-\frac{1}{x}, \text{ or, } \frac{2}{x}=\frac{1}{a}+\frac{1}{b}=\frac{a+b}{ab}$$

$$x=\frac{2ab}{a+b}.$$

Thus, $\frac{2ab}{a+b}$ is the H. M. of a and b.

(c) *To insert p Harmonic means between a and b.*

Insert p Arithmetic means beween $\frac{1}{a}$ and $\frac{1}{b}$ and the reciprocals of these Arithmetic means will be the required Harmonic means.

The required H. M.'s will be found to be

$$\frac{(p+1)ab}{a+pb}, \frac{(p+1)ab}{2a+(p-1)b}, \frac{(p+1)ab}{3a+(p-2)b}, \ldots\ldots \frac{(p+1)ab}{pa+b}.$$

139. Relation between the A. M., G. M., and H. M.

Let A, G, H denote the Arithmetic, Geometric and Harmonic means between a and b respectively; then we have

(i) $A=\frac{a+b}{2}$; (ii) $G=\sqrt{ab}$; (iii) $H=\frac{2ab}{a+b}$.

$$\therefore \qquad A \times H=\frac{a+b}{2} \times \frac{2ab}{a+b}=ab=G^2.$$

$$\therefore \qquad A, G, H \text{ are in G. P.}$$

That is, *the Geometric mean of any two quantities is also the Geometric mean of their Arithmetic and Harmonic means.*

Again, $\qquad A-G=\frac{a+b}{2}-\sqrt{ab}=\frac{1}{2}(\sqrt{a}-\sqrt{b})^2,$

which is positive, if a and b are positive and unequal.

Hence, if a and b are positive and unequal, $A>G$.

Also, since $AH=G^2$ and $A>G$, H must be less than G.

$$\therefore \qquad A > G > H.$$

That is, *the Arithmetic, Geometric and Harmonic means of any two positive and unequal quantities are in descending order of magnitude.*

140. Sum to n terms and to infinity.

There is no *general formula* for the sum to n terms of an H. P.

If the number of terms is infinite, the sum of an H. P. is infinity, so that *Harmonic series are always divergent.*

Example 1. *Find the series in H. P. in which the 7th term is $\frac{1}{20}$ and the 13th term is $\frac{1}{38}$.*

Let a be the first term and b the common difference of the corresponding A. P.; then the 7th and 13th terms of the series in A. P., are 20 and 38 respectively.

Hence we have the 7th term $=a+6b=20$.

and $\qquad$ the 13th term $=a+12b=38$.

whence $\qquad a=2$, and $b=3$.

Hence the series in A. P. is 2, 5, 8, 11,......
and the required series in H. P. is $\frac{1}{2}, \frac{1}{5}, \frac{1}{8}, \frac{1}{11}, \cdots$

Example 2. *If p, q, r, be in A. P., prove that*

$$\frac{qr}{pq+pr}, \frac{rp}{qp+qr}, \frac{pq}{rp+rq}$$

are in H. P. [*Bom. P. E.* 1891]

Since p, q, r are in A. P., we get, dividing by pqr, that

$$\frac{1}{qr}, \frac{1}{rp} \text{ and } \frac{1}{pq} \text{ are also in A. P.}$$

$$\therefore \quad \frac{qr+rp+pq}{qr}, \frac{qr+rp+pq}{rp}, \frac{qr+rp+pq}{pq} \text{ are in A.P.}$$

$$i.e., \quad \frac{pq+pr}{qr}+1, \frac{qp+qr}{rp}+1, \frac{rp+rq}{pq}+1 \text{ are in A. P.}$$

$$\therefore \quad \frac{pq+pr}{qr}, \frac{qp+qr}{rp}, \frac{rp+rq}{pq} \text{ are in A. P.,}$$

and hence their reciprocals are in H. P.

EXERCISE 49.

1. Find the 7th term of the series $\frac{1}{3}+\frac{8}{23}+\frac{4}{11}+\cdots$

2. Continue the H. P. 4, 5, $6\frac{2}{3}$,......each way to 4 terms.

3. Find the nth term of the series $4+4\frac{2}{7}+4\frac{8}{13}+5+\ldots\ldots$ [Cal. F.A. 1886]

4. The 12th term of an H.P. is $\frac{1}{5}$ and the 19th term is $\frac{3}{22}$. Find the 4th term.

5. If the pth term of an H. P. is equal to q and the qth term is equal to p, prove that the $(p+q)$th term is equal to $\frac{pq}{p+q}$.

6. Find the mth term of an H. P. whose first term is a, whose last term is c, and whose number of terms is n. [Mad. F. A. 1884]

7. If the mth term of an H.P. is n, and the nth term is m, then the rth term is $\frac{mn}{r}$. [All. I. E. 1909]

8. If x, y, z be the pth, qth and rth terms respectively of an H. P., show that $(q-r)yz+(r-p)zx+(p-q)xy=0$.

9. Find the Harmonic mean between

(i) 4 and 7; (ii) 9 and 19; (iii) $x+y$ and $x-y$.

10. Insert

(i) 4 Harmonic means between 1 and 30. [All. I. E. 1892]

(ii) 3 Harmonic means between $a+b$ and $a-b$.

11. If a_1, a_2 be the A. M's., h_1, h_2 the H. M's., and g_1, g_2 the G.M's, between a and b, show that $a_1h_2=a_2h_1=g_1g_2=ab$. [Mad. F. A. 1891]

12. If a, b, c be in G. P. and if p be the A. M. between a and b, and q be the A. M. between b and c, then b will be the H. M. between p and q. [Cal. F. A. 1893]

13. If a, b, c be three quantities such that a is the A. M. between b and c, and c is the H. M. between a and b, show that b is the G. M. between a and c; and compare a, b, c. [All. I. E. 1899]

14. If $2(y-a)$ is an H. M. between $y-x$ and $y-z$, then $x-a, y-a, z-a$ form a G. P. [All. I. E. 1890]

15. If a, b, c are in A. P. and b, c, d in H. P. prove that $a:b=c:d$.

16. If a^2, b^2, c^2 are in A. P., show that $b+c$, $c+a$ and $a+b$ are in H. P. [All I.E. 1927]

17. If $\dfrac{a-x}{px}=\dfrac{a-y}{qy}=\dfrac{a-z}{rz}$, and p, q, r be in A. P., show that x, y, z are in H. P. [Bom. P. E. 1893]

18. The quantities x, y, z are in A. P. They are also in H. P. Prove that they are in G. P. [Cal. F.A. 1875]

19. If a, b, c be in A. P., b, c, d in G. P., and c, d, e in H. P., prove that a, c, e are in G. P. [Bom. P. E. 1886]

20. If the roots of the equation $a(b-c)x^2+b(c-a)x+c(a-b)=0$ be equal, show that a, b, c are in H. P. [All. I. E. 1926]

CHAPTER XIII

MISCELLANEOUS THEOREMS AND PRINCIPLES.

1. Undetermined Co-efficients.

141. Principle of Undetermined Co-efficients.

If $\quad a_0+a_1x+a_2x^2+ \quad \cdots \quad \cdots \quad \cdots \quad +a_nx^n$

$\quad =b_0+b_1x+b_2x^2+ \quad \cdots \quad \cdots \quad \cdots \quad +b_nx^n$

for all values of x, that is to say, if the two expressions are identically equal, then

$$a_0 = b_0,\ a_1 = b_1,\ a_2 = b_2, \quad \cdots \quad \cdots \quad a_n=b_n,$$

that is to say, the co-efficients of like powers of x on either side are equal each to each.

Since

$$a_0+a_1x+\cdots+a_nx^n=b_0+b_1x+\cdots+b_nx^n$$

for all values of x, the equality will hold if $x=0$.

Putting $x=0$ on either side, we get $a_0=b_0 \quad \cdots \quad (1)$

Hence, we have now

$$a_1x+a_2x^2+\cdots+a_nx^n=b_1x+b_2x^2+\cdots+b_nx^n.$$

Dividing both sides by x, we have

$$a_1+a_2x+\cdots+a_nx^{n-1}=b_1+b_2x+\ldots+b_nx^{n-1}$$

Again, putting $x=0$, we get $a_1=b_1 \quad \cdots \quad (2)$

Hence, we have now

$$a_2x+a_3x^2+\cdots\cdots+a_nx^{n-1}=b_2x+b_3x^2+\cdots+b_nx^{n-1}.$$

Again, dividing both sides by x and then putting $x=0$, we have $\quad a_2 = b_2 \quad \cdots \quad \cdots \quad (3)$

Proceeding in this manner, we see that the co-efficients of like powers of x on either side are equal each to each.

This is known as *the Principle of Undetermined co-efficients.*

This principle is also true in the case when the identical relation involves more than one variable ; and in that case, *the co-efficients of like terms on either side will be equal each to each.*

Cor. From the above theorem it follows that if $a_0+a_1x+a_2x^2+\cdots+a_nx^n=0$ for all values of x, then $a_0=a_1=a_2=\cdots=a_n=0$, *i.e.*, all the co-efficients must separately vanish.

We work out below some examples which show how the Principle of Undetermined Co-efficients may be usefully employed for various purposes.

Example 1. *Express* $2x^3-3x^2+4x+2$ *in descending powers of* $x-1$.

Since the given expression is of the third degree we assume

$$\begin{aligned}2x^3-3x^2+4x+2&=A(x-1)^3+B(x-1)^2+C(x-1)+D\\&=Ax^3-(3A-B)x^2+(3A-2B+C)x-(A-B+C-D)\end{aligned}$$

where A, B, C, D are constants to be determined.

Hence, by the principle of undetermined co-efficients, comparing and equating co-efficients of like powers of x on either side, we have

$$A=2,\ 3A-B=3,\ 3A-2B+C=4, \text{ and } A-B+C-D=-2,$$

whence $A=2$, $B=3$, $C=4$ and $D=5$.

Hence, $2x^3-3x^2+4x+2=2(x-1)^3+3(x-1)^2+4(x-1)+5$.

Example 2. *For what value of k is the expression* $x^2+2xy-15y^2-10x-2y+k$ *the product of linear factors?*

Taking the terms of the second degree $x^2+2xy-15y^2$, we see that they can be resolved into factors $(x-3y)(x+5y)$.

Hence the given expression must be a product of the form

$$(x-3y+a)(x+5y+b)$$

which, on multiplication, becomes

$$x^2+2xy-15y^2+(a+b)x+(5a-3b)y+ab.$$

Since the given expression is to be identically equal to the above expression, we have, by comparing and equating the co-efficients of like terms, by the principle of undetermined co-efficients,

$$a+b=-10,\ 5a-3b=-2,\ ab=k.$$

From the first two of these equations, $a=-4$, $b=-6$.

$\therefore$ $k=ab=24$, the required value.

II. Mathematical Induction.

142. Mathematical Induction.

Mathematical Induction is the name given to a method of establishing the truth of a theorem generally from a knowledge of its truth in particular cases.

The method may be thus described :

Assuming the truth of a theorem in any one case, whatever that case may be, we prove its truth in another case which may be regarded as the next case. Then by trial we prove the theorem in some particular case ; hence the theorem will hold in the next case : hence in the next case after that : and so universally.

It should be remarked that this process is not really induction, in the sense in which that term is used in Logic. The method of reasoning involved is strictly a deductive one.

An example is given below to illustrate the application of this method of proof.

Example 1. *Prove by induction that*

$$1^2+2^2+3^2+\cdots \textit{to } n \textit{ terms}=\tfrac{1}{6}n(n+1)(2n+1)$$

[*Bom. P. E.* 1886.]

Let us assume the truth of this theorem for r terms. That is to say, we assume that

$$1^2+2^2+3^2+\ldots+r^2=\tfrac{1}{6}r(r+1)(2r+1).$$

Hence, adding $(r+1)^2$ to both sides, we have

$$\begin{aligned}1^2+2^2+3^2+\ldots+r^2+(r+1)^2&=\tfrac{1}{6}r(r+1)(2r+1)+(r+1)^2\\&=\tfrac{1}{6}(r+1)\{r(2r+1)+6(r+1)\}\\&=\tfrac{1}{6}(r+1)(2r^2+7r+6)\\&=\tfrac{1}{6}(r+1)(r+2)(2r+3)\\&=\tfrac{1}{6}(r+1)\{(r+1)+1\}\{2(r+1)+1\}.\end{aligned}$$

Thus, if the theorem is true for r terms, it is true for $(r+1)$ terms.

But the theorem is actually true when $n=1$, 2, or 3 ; for

$$1^2=\tfrac{1}{6}.\ 1\ (1+1)(2+1)\ ;$$
$$1^2+2^2=\tfrac{1}{6}.\ 2\ (2+1)(2.2+1)\ ;$$
$$1^2+2^2+3^2=\tfrac{1}{6}.\ 3(3+1)(2.3+1).$$

Hence, the theorem is true when $n=4$; hence, when $n=5$; hence, when $n=6$; and so on.

Thus, the theorem is universally true.

III. Convergency and Divergency of Series.

143. Definitions.

We have already met with series in Geometrical Progression which are such that though the number of terms of the series is infinite, their sum never exceeds a finite quantity. Thus for instance :

$$1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4} \quad \dots \quad \dots \quad ad\ inf.=2.$$

On the other hand, series in Arithmetical or Harmonical Progression are always such that if an infinite number of terms is taken, the sums of the series become infinite.

Thus we have, broadly speaking, two types of infinite series : the former type is called *convergent*, and the latter *divergent*.

Formally we can define the distinction thus : An infinite series is said to be *convergent* when the sum of the first n terms of the series cannot be made to exceed numerically some finite quantity, however large n may be ; and *divergent* when the sum of the first n terms of the series can be made to exceed numerically any finite quantity however large, by taking n large enough.

144. Test of convergency when series summable.

Whether a series is convergent or otherwise can be easily determined in cases where the sum of the series to n terms can be found as a function of n. (The student is warned however that such an expression for the sum to n terms cannot always be found ; a very familiar case is that of a series in Harmonical Progression.)

In such a case, we have only to see to what value this function of n approaches as n is made infinitely large ; if the function tends to a finite limit the series is convergent ; if the function tends to become infinite, the series is divergent.

For instance, in the G. P.

$$1+r+r^2+r^3+\dots$$

the sum to n terms is

$$\frac{1-r^n}{1-r} \text{ when } r<1, \text{ and } \frac{r^n-1}{r-1} \text{ when } r>1.$$

When n becomes infinite,

$$\frac{1-r^n}{1-r} \text{ becomes } \frac{1}{1-r}, \text{ since } r^{\infty}=0 \text{ when } r<1;$$

but $$\frac{r^n-1}{r-1} \text{ becomes infinite, since } r^{\infty}=\infty \text{ when } r>1.$$

[See Art. 11]

Thus, the given Geometric series is convergent when $r<1$, and divergent when $r>1$. If $r=1$, the series becomes $1+1+1\cdots$ *ad inf.*, and is obviously divergent.

145. Importance of testing Convergency.

For practical purposes, an infinite series which is divergent, *i.e.*, which does not tend to any finite limit, is useless; so that it is very necessary whenever any infinite series is in question, to enquire whether it is convergent, or if the series involves a variable, then for what value or values of the variable the series is convergent; just as we have ascertained in the previous article that an infinite G. P. is convergent when $r<1$, and divergent otherwise.

But not merely is a divergent series useless, but when a function is expanded in the form of an infinite series and that series happens to be divergent, we may be led to the most absurd results, if the equality between the function and the series is assumed to hold.

Thus, by ordinary division, we can prove that

$$\frac{1}{1-x}=1+x+x^2+x^3+\cdots\cdots\cdots ad\ inf.$$

If $x<1$, the series on the right is an infinite G. P. with common ratio less than unity, and hence convergent, and its sum is therefore actually equal to the expression on the left.

But if $x>1$, the series on the right obviously has its sum infinite, while the function on the left is finite. Thus, *e.g.*, if $x=2$, we get

$$\frac{1}{1-2}=-1=1+2+2^2+2^3+\cdots\cdots\cdots ad\ inf.$$

an obviously absurd result.

The real reason of such an absurd anomaly is this, that the equality does not really hold in such cases. The fact is that when any function of a variable quantity is expanded in terms of the variable in the form of an infinite series,

the series thus obtained may or may not be equal to the function for all values of the variable. The series is equal to the function for those values of the variable which render the series convergent, and not equal for other values of the variable. Hence we see the importance of testing the convergency of an infinite series.

146. General Convergency Test.

An infinite series is convergent, if after some fixed term the ratio of each term to the preceding one is numerically less than some fixed quantity which is itself less than unity.

Let $u_1+u_2+u_3+u_4+\cdots\cdots$ be the series after the fixed term, and suppose that the terms of the series are such that $\frac{u_2}{u_1}<r, \frac{u_3}{u_2}<r, \frac{u_4}{u_3}<r, \cdots\cdots$ where r is itself less than unity.

Then $u_1+u_2+u_3+u_4+\cdots\cdots\cdots\cdots$

$$=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_1}+\frac{u_4}{u_1}+\cdots\cdots\cdots\right)$$

$$=u_1\left(1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\cdots\right)$$

$$<u_1(1+r+r^2+r^3+\cdots\cdots\cdots\cdots)$$

i.e., $$<\frac{u_1}{1-r}, \text{ since } r<1.$$

Now since $\frac{u_1}{1-r}$ is a finite quantity, and the sum of the series after the fixed term is less than the finite quantity, the given series, and hence the whole series, must be convergent.

Example 1. *Is the following series convergent*:

$$\frac{1}{1}+\frac{3}{2}+\frac{3^2}{3}+\frac{3^3}{4}+\cdots\cdots\cdots+\frac{3^{n-1}}{n}+\cdots?$$

Let u_n denote the nth term. Then

$$u_n=\frac{3^{n-1}}{n}, \text{ and } u_{n+1}=\frac{3^n}{n+1}.$$

Hence $\frac{u_{n+1}}{u_n}=\frac{3n}{n+1}$, which is >1, when $n>1$.

Therefore the series is not convergent.

CHAPTER XIV

PERMUTATIONS AND COMBINATIONS.

147. Definitions.

The different *orders of arrangement* that can be made out of a number of things, by taking some out of them or all of them together, are called their *Permutations.*

Thus, if there are three things denoted by the letters a, b, c, then their different permutations, taking two things at a time, are : ab, ba, bc, cb, ca, ac.

If we take these three things all together, the different permutations are : abc, acb, bca, bac, cab, cba.

The number of permutations of n different things taken r at a time is usually denoted by the symbol ${}^nP_r$.

The different collections or groups that can be made out of a number of things, irrespective of the order of their arrangement, by taking some out of them or all of them ~~together,~~ are called their *Combinations.*

Of course, it is evident that if all the things are taken together, they can form only one combination.

Thus, if there are three things denoted by the three letters a, b, c, then their different combinations, taking two things at a time, are : ab, bc, ca.

If we take the three things all together there can be but one combination : abc.

The number of combinations of n different things taken r at a time is usually denoted by the symbol ${}^nC_r$.

It is to be particularly noted that the difference between permutations and combinations lies in this that permutation mean different orders of arrangement, while combin tions mean only groups or collections irrespective of th order of arrangement.

Hence in any problem relating to permutations and co-binations, the student will have carefully to see whether it involves any order of arrangement, or whether it merely involves grouping. In the former case it is a problem in permutation; in the latter case it is a problem in combination.

Thus, to find out the number of words that can be formed out of a certain number of letters involves orders of arrangement, since a different arrangement means a different word; so it is a problem in permutation. But to find out how many different sums can be made out of a number of coins by selecting some of these involves only grouping for here order of arrangement is immaterial; hence it is a problem in combination.

I. Permutations.

148. An Important Principle.

If there are m *ways of performing one operation, and when it has been performed in any one of these ways, there are* n *ways of performing a second operation, the total number of ways of performing the two operations will be* mn.

Let us first take a particular example.

Suppose there are 5 different routes from A to B, and 7 different routes from B to C. In how many ways can a person go from A to C through B?

Since there are 5 different routes from A to B, the person can go from A to B in 5 different ways, and since there are 7 different routes from B to C corresponding to each of the 5 ways of going from A to B, there are 7 ways of going from B to C. Hence it is obvious that altogether the number of ways in which the person can go from A to C through B is 5×7, *i.e.*, 35.

Similarly, since there are m ways of performing the first operation, and corresponding to each of the m ways there are n ways of performing the second operation, the total number of ways in which both the operations can be performed must be $m\times n$.

This principle may be extended to the case of any number of operations. Thus, if corresponding to each of the above $m\times n$ ways of performing the two operations there are p ways of performing a third operation, the total number of ways in which all the three operations may be performed will be $m\times n\times p$; and so on.

149. General Permutation Formula.

(*i*) *To find the number of permutations of* n *different things taken* r *at a time.*

Let $a, b, c, \cdots p$ be the n different things.

Then, if we suppose that there are r blank places, the number of permutations of n things taken r at a time is the same as the number of ways in which we can fill up the r blank places, putting one in each position.

Now, the first place may be filled up in n different ways, for it may be filled up by *any one* of the n things; *i.e.*, ${}^nP_1 = n$.

After filling up the first place by any one of the n things, we can fill up the second place by any one of remaining $n-1$ things in $n-1$ different ways; and since each way of filling up the first place can be associated with each way of filling up the second, the total number of ways in which the first two places can be filled up is $n(n-1)$; *i.e.*, ${}^nP_2 = n(n-1)$.

And after filling up the first two places by any two of the n things, we can fill up the third place by any one of the remaining $n-2$ things in $n-2$ different ways; and therefore reasoning as before, the total number of ways in which the first three places can be filled up is $n(n-1)(n-2)$; *i.e.*, ${}^nP_3 = n(n-1)(n-2)$.

Proceeding thus, and observing that at any stage the number of factors in the result is the same as the number of places filled up and that these factors begin from n and successively diminish by 1, we have the total number of ways of filling up the r places

$$= n(n-1)(n-2)(n-3)\ldots\ldots \text{to } r \text{ factors}$$
$$= n(n-1)(n-2)(n-3)\ldots\ldots \{n-(r-1)\}$$
$$= n(n-1)(n-2)(n-3)\ldots\ldots(n-r+1)$$

i.e., ${}^nP_r = n(n-1)(n-2)(n-3)\ldots\ldots(n-r+1)$.

Alternative method:

Let $a, b, c, \ldots\ldots p$ be the n different things.

Then the permutations of the n things taken r at a time in which a stands first are obtained by taking the permutations of the $n-1$ things $b, c, \ldots\ldots p$ taken $r-1$ at a time and placing a before each of them. Hence the number of permutations of the n things taken r at a time in which a stands first is equal to ${}^{n-1}P_{r-1}$. Similarly the number of permutations of n things taken r at a time in which b

stands first is equal to $^{n-1}P_{r-1}$; and so on. Since there are n things, the total number of the permutations is equal to the sum of all these n groups of permutations and therefore we have

$$^{n}P_r = n \times {}^{n-1}P_{r-1} \;;$$

similarly,
$$^{n-1}P_{r-1} = (n-1) \times {}^{n-2}P_{-2} \;;$$
$$^{n-2}P_{r-2} = (n-2) \times {}^{n-3}P_{r-3} \;;$$

$$^{n-r+2}P_2 = (n-r+2) \times {}^{n-r+1}P_1 \;;$$

and
$$^{n-r+1}P_1 = (n-r+1).$$

Multiplying together the vertical columns and cancelling the common factors from both sides, we have

$$^{n}P_r = n(n-1)(n-2)(n-3)\cdots\cdots(n-r+1).$$

(*ii*) *To find the number of permutations of* n *different things taken all together.*

Since $\quad {}^{n}P_r = n(n-1)(n-2)\cdots\cdots(n-r+1),$

putting $\quad r = n$, we have

$$^{n}P_n = n(n-1)(n-2)\cdots\cdots(n-n+1)$$
$$= n(n-1)(n-2)\cdots\cdots 2.1,$$

i.e., the continued product of the first n natural numbers.

150. Factorial Notation

The continued product of the first n natural numbers, namely, $1.2.3\cdots n$ is usually denoted by the symbol $\underline{|n}$ or $n\,!$ which is read *factorial n*. Thus,

$$\underline{|1} = 1 \;; \qquad \underline{|2} = 1.2 \;; \qquad \underline{|3} = 1.2.3 \;; \text{ etc}$$

Obviously $\underline{|n} = 1.2.3.\cdots\cdots n = {}^{n}P_n.$

Again, $\quad \underline{|n} = {}^{n}P_n = n(n-1)(n-2)\cdots\cdots\cdots 2.1 = {}^{n}P_{n-1}.$

Also, $\quad \underline{|n} = n\underline{|n-1} = n(n-1)\underline{|n-2} = n(n-1)(n-2)\underline{|n-3} = \ldots$
$$= n(n-1)(n-2)\cdots\cdots(n-r+1)\underline{|n-r}.$$

Using the factorial notation, we may express the formulæ for $^{n}P_r$ in a more concise form. Thus

$$^{n}P_r = n(n-1)(n-2)\cdots\cdots(n-r+1)$$
$$\frac{n(n-1)(n-2)\ldots\ldots(n-r+1)\underline{|n-r}}{\underline{|n-r}}$$

$$\frac{\underline{|n}}{\underline{|n-r}}$$

Putting $r=0$ in (1) we have ${}^nP_0=\dfrac{\lfloor n}{\lfloor n}=1.$

Putting $r=n$ in (1) we have ${}^nP_n=\dfrac{\lfloor n}{\lfloor n-n}=\dfrac{\lfloor n}{\lfloor 0}.$

But ${}^nP_n=\lfloor n$ $\qquad\therefore\quad \lfloor 0=1.$

Example 1. *If* ${}^nP_6=30\times{}^nP_4$, *find* n.

By the question, we have

$$n(n-1)(n-2)(n-3)(n-4)(n-5)=30n(n-1)(n-2)(n-3)$$

$\therefore$ $\qquad (n-4)(n-5)=30$

or, $\qquad n^2-9n+20=30$

or, $\qquad n^2-9n-10=0$

or. $\qquad (n-10)(n+1)=0$

$\therefore$ $\qquad n=10$ or -1.

Since n is necessarily a positive integer, we have $n=10$, and the negative value is inadmissible.

Example 2. *Prove that* ${}^{2n}P_n=2^n.\{1.3.5\ldots(2n-1)\}$.

$$\begin{aligned}{}^{2n}P_n&=\frac{\lfloor 2n}{\lfloor 2n-n}=\frac{1.2.3\ldots 2n}{\lfloor n}\\&=\frac{\{1.3.5\ldots(2n-1)\}(2.4.6\ldots 2n)}{1.2.3\ldots n.}\\&=\frac{\{1.3.5\ldots(2n-1)\}2^n(1.2.3\ldots n)}{1.2.3\ldots n}\\&=2^n\{1.3.5\ldots(2n-1)\}.\end{aligned}$$

Example 3. *How many words can be formed, using all the letters of the word* Cambridge ?

The given word contains altogether 9 letters and the number of words required is evidently the same as the permutations of 9 things taken all together.

Hence the required number of words

$={}^9P_9=9.8.7.6.5.4.3.2.1=362880.$

Example 4. *How many numbers lying between* 2000 *and* 5000 *can be formed with the digits* 1, 3, 4, 7, 9, 0, *no digit being used more than once in each number ?*

The numbers will evidently consist of 4 digits and will begin with either 3 or 4. Since there are 6 digits to be used, the numbers which begin with 3 are obtained by taking the permutations of the remaining 5 digits 1, 4, 7, 9, 0 taken 3 at a time, and therefore their number $= {}^5P_3 = 5 \times 4 \times 3 = 60$.

Similarly, the number of those beginning with $4 = {}^5P_3 = 60$.

Hence the total number required $= 60 + 60 = 120$.

Example 5. *In an office there are* 8 *different situations vacant, and there are* 15 *candidates of which* 10 *are Hindus and the rest Mahomedans. Three of the situations are to be held by Hindus,* 2 *by Mahomedans, and the rest by either Hindus or Mahomedans. In how many ways can these be filled up ?*

Since there are 10 Hindus and 3 of the situations are to be filled up by Hindus, this can be done in ${}^{10}P_3$ ways. Similarly, since there are 5 Mahomedans, 2 of the situations can be filled up by Mahomedans in ${}^5P_2$ ways. After these 5 situations are filled up, the remaining 3 situations are to be filled up by the remaining 10 candidates and this can be done in ${}^{10}P_3$ ways. Hence the total number of ways in which all the 8 situations can be filled up

$$= {}^{10}P_3 \times {}^5P_2 \times {}^{10}P_3 = 720 \times 20 \times 720 = 10368000.$$

151. Restricted Permutations.

In many problems on Permutation restrictions are put upon the positions of the things or upon the orders of arrangement ; for instance, some things are always to be kept together, or some things are not to be kept together, or some things are to be placed in definite positions, and so on. The following examples will show how such problems are solved.

Example 1. *How many words can be formed by using all the letters of the word* Cambridge, *so that all the vowels may remain together ?*

There are 9 letters in all, and the vowels are 3. If we take the 3 vowels together and consider them as *one*, the number of letters becomes 7, *viz.*, $c, m, b, r, d, g, (a\ i\ e)$.

Hence these 7 letters can be arranged in $\underline{|7}$ ways.

But the 3 vowels *a, i, e.*, while keeping together, may themselves be arranged in $\underline{|3}$ ways ; and each of the former $\underline{|7}$ ways may be combined with each of the latter $\underline{|3}$ ways.

Hence the total number of words that can be formed with all the vowels kept side by side

$$=\underline{|7}\times\underline{|3}=7.6.5.4.3.2.1.\times 3.2.1=30240.$$

Example 2. *There are four works each of 3 volumes and three works each of 4 volumes. In how many ways can these 24 books be arranged on a shelf so that the volumes of the same work are not separated ?*

Since there are 7 works altogether they can be arranged, taking the volumes of each work as forming one group, in ${}^7P_7$, *i.e.*, $\underline{|7}$ ways.

But the 3 volumes of each of the four works can be arranged among themselves without being separated in ${}^3P_3$ *i.e.*, $\underline{|3}$ ways, and the 4 volumes of each of the three works in ${}^4P_4$, *i.e.*, $\underline{|4}$ ways. Hence the total number of ways of arranging the books under the given condition

$$=\underline{|7}\times\underline{|3}\times\underline{|3}\times\underline{|3}\times\underline{|3}\times\underline{|4}\times\underline{|4}\times\underline{|4}=\underline{|7}\times(\underline{|3})^4\times(\underline{|4})^3.$$

EXERCISE 50.

1. Find the values of :

(i) ${}^{12}P_5$. (ii) ${}^{10}P_{10}$. (iii) ${}^{15}P_r\ (r<15)$.

2. Prove the following identities :

(i) $\underline{|100}=2^{50}.\ \underline{|50}(1.3.5\ldots\ldots99)$.

(ii) $\underline{|2n}=2^n.\ \underline{|n}\{1.3.5\cdots\cdots\cdots(2n-1)\}$

(iii) $(n+1)(n+2)(n+3)\cdots$to n factors$=2.6.10.14\cdots$to n factors.

3. If ${}^nP_5=60\times{}^{n-1}P_3$, find n.

4. If ${}^nP_4=840$, find n.

5. If ${}^{n+1}P_6:{}^{n-1}P_7=5:12$, find n.

6. If ${}^{10}P_r=5040$, find r.

7. Without assuming the formula for ${}^nP_r$, prove that ${}^nP_r={}^{n-1}P_r+r.{}^{n-1}P_{r-1}$.

8. Prove that ${}^nP_r=(n-r+1)\times{}^nP_{r-1}$, and hence establish the formula for ${}^nP_r$.

9. Prove that

$${}^nP_n=1+1.{}^1P_1+2.{}^2P_2+3.{}^3P_3+\cdots\cdots+(n-1)\times{}^{n-1}P_{n-1}.$$

10. Find the number of permutations of the letters of the word *Nares* taken all together. [Cal. Int. 1918]

11. There are 12 competitors in a race for 3 prizes; in how many ways can the prizes be given?

12. Two persons go into a railway carriage where there are six vacant seats. In how many different ways can they seat themselves? [Cal. Int. 1910]

13. How many changes can be rung with 6 bells? How many of these will begin with one particular bell?

14. In how many ways can the letters of the word *Orange* be re-arranged? How many of these will begin with *O*?

15. How many numbers lying between 100 and 1000 can be formed with the digits 2, 3, 4, 5, 6, and 7?

16. How many numbers lying between 3000 and 4000 can be formed with the digits 1, 2, 3, 4, 5, 6, and 7?

17. How many numbers each lying between 1000 and 10000 can be formed with the digits 0, 1, 3, 5, 7, 9?

18. Of the permutations of the letters of the word *Logarithm* taken all together, how many do not begin with *th*?

19. If words be formed with the letters of the word *Question*, how many will begin with *q* and end with *n*?

20. How many words can be formed with the letters of the word *Equation*, so that the consonants should always occupy the odd places?

21. Find how many words can be formed of the letters of the word *Failure*, the four vowels always coming together. [Cal. Int. 1940]

22. There are *m* men and *n* monkeys, *n* being greater than *m*. Find the number of ways in which each man may become the owner of one monkey. [Bom. P. E. 1891]

23. If 8 men and 5 women apply for 5 different situations, 3 of which must be filled by men and 2 by women, in how many ways can the situations be filled?

24. Find the number of ways in which the letters of the word *Courage* may be re-arranged (i) without changing the order of the vowels; (ii) without changing the place of any vowel; (iii) without changing the relative order of vowels and consonants.

25. Find the number of different arrangements that can be made of bars of the seven prismatic colours so that the blue and the green shall never come together. [Bom. P.E. 1885]

26. Find the number of words beginning with a consonant and ending with a vowel which can be formed out of the word *Emulation*.

27. In how many of the permutations of 10 things taken 4 at a time will one particular thing (i) always occur, and (ii) never occur? [Cal. Int. 1936]

28. If numbers are formed using all the digits 1, 2, 3, 4, 5, 6, 7, 8, 9, how many of them are odd and how many even?

29. Find the number of numbers less than 1000 and divisible by 5 which can be formed with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, each digit not occurring more than once in each number. [Cal. Int. 1942]

30. If there are m works each of a volumes and n works each of b volumes, show that the number of ways in which all the books may be arranged on a shelf, so that the volumes of the same work are not separated is equal to $(\lfloor a)^m (\lfloor b)^n \lfloor m+n$.

31. Show that the number of ways in which n books may be arranged on a shelf so that two particular books shall not be together is $(n-2)\lfloor n-1$.

32. Show that the number of permutations of n things taken all together in which r particular things are to be in an assigned order is $\lfloor n \div \lfloor r$.

II. Combinations

General Combination Formula.

To find the number of combinations of n *different things taken* r *at a time.*

Let ${}^nC_r$ denote the required number. Then since every combination of r different things produces $\lfloor r$ permutations when the r things are arranged in all possible ways, the ${}^nC_r$ combinations, each combination containing r different things, will produce ${}^nC_r \times \lfloor r$ permutations, and this is obviously the total number of permutations of n different things taken r at a time which is equal to $n(n-1)(n-2)\cdots\cdots(n-r+1)$. [Art. 149.]

Hence, we have

$$^nC_r \times \lfloor r = n(n-1)(n-2)......(n-r+1)$$

$$\therefore \qquad ^nC_r = \frac{n(n-1)(n-2)......(n-r+1)}{\lfloor r}.$$

Using the factorial notation, the above results may also be expressed in the following way :

Multiplying both the numerator and the denominator of the expression for ^nC_r by $\lfloor n-r$, we have

$$^nC_r = \frac{n(n-1)(n-2)\cdots\cdots(n-r+1) \times \lfloor n-r}{\lfloor r \, \lfloor n-r}$$

$$= \frac{\lfloor n}{\lfloor r \, \lfloor n-r}.$$

Independent Proof (without assuming the formula for Permutation) :

Let the n different things be denoted by the n different letters $a, b, c, d, ...$; and let ^nC_r be the number of combinations of n things taken r at a time.

Then, of the combinations of n letters taken r at a time, those combinations which contain the letter a are obtained by combining a with each of the combinations of the remaining $n-1$ letters $b, c, d, \cdots\cdots$ taken $r-1$ at a time ; that is, there are as many combinations of n letters taken r at a time each containing a particular letter a, as there are combinations of $n-1$ letters taken $r-1$ at a time. Hence, the number of combinations which contain a is $^{n-1}C_{r-1}$; similarly, the number of those which contain b is $^{n-1}C_{r-1}$; and so on for each of the n letters.

Hence, if all the combinations of the n letters taken r at a time be written down, we find that every one of the n letters appears in them $^{n-1}C_{r-1}$ times and therefore the total number of letters written down is $n \times {}^{n-1}C_{r-1}$. But it is also evident that the total number of letters written down is equal to $r \times {}^nC_r$ for there are ^nC_r combinations each containing r letters.

Hence, $\qquad r \times {}^nC_r = n \times {}^{n-1}C_{r-1}$.

Since this relation is true for all values of n and r, r being not greater than n, we have

$$^{n}C_r = \frac{n}{r} \times {}^{n-1}C_{r-1}$$

$$^{n-1}C_{r-1} = \frac{n-1}{r-1} \times {}^{n-2}C_{r-2}$$

$$^{n-2}C_{r-2} = \frac{n-2}{r-2} \times {}^{n-3}C_{r-3},$$

....

$$^{n-r+2}C_2 = \frac{n-r+2}{2} \times {}^{n-r+1}C_1$$

and $$^{n-r+1}C_1 = \frac{n-r+1}{1}.$$

Now, multiplying together the vertical columns and cancelling the common factors from each side, we have

$$^{n}C_r = \frac{n(n-1)(n-2)\cdots\cdots(n-r+1)}{1.2.3\cdots\cdots r}.$$

153. Complementary Combinations.

To prove that the number of combinations of n *things taken* r *at a time is equal to the number of combinations of* n *things taken* n − r *at a time.*

Since $^{n}C_r = \frac{\lfloor n}{\lfloor r \, \lfloor n-r}$, we have, on changing r into $n-r$,

$$^{n}C_{n-r} = \frac{\lfloor n}{\lfloor n-r \, \lfloor n-(n-r)} = \frac{\lfloor n}{\lfloor n-r \, \lfloor r}.$$

Hence $$^{n}C_r = {}^{n}C_{n-r}.$$

Direct proof (without assuming any formula):

When one of the combinations of n things taken r at a time is formed, there remain $n-r$ things which may be considered as forming one of the combinations of the n things taken $n-r$ at a time. Hence there must be as many combinations of n things taken $n-r$ at a time as there are combinations of n things taken r at a time.

Hence, $$^{n}C_r = {}^{n}C_{n-r}.$$

Two such groups of r things and the remaining $n-r$ things are called *complementary combinations.*

Cor. 1. From the above reasoning it is evident that $^{n}C_r$ is also the number of ways in which n things may be divided into two groups of r and $n-r$ things respectively.

Cor. 2. Since ${}^nC_r = {}^nC_{n-r}$, putting $r=0$, we have

$$ {}^nC_0 = {}^nC_{n-0} = {}^nC_n = 1. $$

Cor. 3 *If* ${}^nC_x = {}^nC_y$, *then either* $x=y$, or $x+y=n$.

154. To prove that ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.

$$ {}^nC_r + {}^nC_{r-1} = \frac{|\underline{n}}{|\underline{r}\ |\underline{n-r}} + \frac{|\underline{n}}{|\underline{r-1}\ |\underline{n-r+1}} $$

$$ = \frac{|\underline{n}}{|\underline{r-1}\ |\underline{n-r}}\left\{\frac{1}{r} + \frac{1}{n-r+1}\right\} $$

$$ = \frac{|\underline{n}}{|\underline{r-1}\ |\underline{n-r}} \cdot \frac{n+1}{r(n-r+1)} $$

$$ = \frac{|\underline{n+1}}{|\underline{r}\ |\underline{n-r+1}} $$

$$ = {}^{n+1}C_r. $$

Alternative proof :

${}^nC_r$ = the number of combinations of $n+1$ things taken r at a time which do not contain a specified thing ; and

${}^nC_{r-1}$ = the number of combinations of $n+1$ things taken r at a time which contain that specified thing.

But the sum of these two must be the total number of combinations of $n+1$ things r at a time.

Hence, $\quad {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r.$

Example 1. *Find the values of* ${}^{24}C_5$ *and* ${}^{30}C_{27}$.

$$ {}^{24}C_5 = \frac{24\cdot23\cdot22\cdot21\cdot20}{1\cdot2\cdot3\cdot4\cdot5} = 4\times23\times22\times21 = 42504. $$

$$ {}^{30}C_{27} = {}^{30}C_3 = \frac{30\cdot29\cdot28}{1\cdot2\cdot3} = 5\times29\times28 = 4060. $$

Example 2. *If* $m = {}^nC_2$, *prove that* ${}^mC_2 = 3\times{}^{n+1}C_4$.

[*Cal. Int.* 1912

Here, $\quad m = {}^nC_2 = \dfrac{n(n-1)}{2}.$

$$ \therefore \quad m-1 = \frac{n(n-1)}{2} - 1 = \frac{n^2-n-2}{2} = \frac{(n+1)(n-2)}{2}. $$

$$ \text{Hence,} \quad {}^mC_2 = \frac{m(m-1)}{2} = \frac{1}{2}\left\{\frac{n(n-1)}{2}\cdot\frac{(n+1)(n-2)}{2}\right\} $$

$$ = 3\times\frac{(n+1)n(n-1)(n-2)}{1.2.3.4} = 3\times{}^{n+1}C_4. $$

Example 3. *A party of* 5 *has to be formed out of* 12 *men. How many different selections can be made ?*

The required number $= {}^{12}C_5 = \frac{12.11.10.9.8}{1.2.3.4.5} = 11 \times 9 \times 8 = 792.$

Example 4. *How many numbers are there consisting of five different digits arranged in ascending order of magnitude ?*

Since of all the numbers that can be formed with any set of five different digits there is only one number in ascending order of magnitude, and since there is no number possible beginning with 0, we shall only consider the remaining 9 digits, and therefore the total number of the required numbers

$$= {}^9C_5 = {}^9C_4 = \frac{9.8.7.6}{1.2.3.4} = 126.$$

Example 5. *How many triangles can be formed by joining the vertices of a figure of* n *straight lines as sides, and how many diagonals has it ?* [*Cal. F. A.* 1867]

Since a triangle is formed by joining any three vertices, and since the figure has n vertices, the required number of triangles $= {}^nC_3 = \frac{n(n-1)(n-2)}{1.2.3}$.

Again, the number of straight lines obtained by joining the n vertices two by two $= {}^nC_2 = \frac{n(n-1)}{1.2}$.

But this number includes the n sides of the figure.

Hence the required number of diagonals

$$= \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}.$$

Example 6. *Find the number of different straight lines obtainable by joining* n *different points, no three of which are collinear, with the exception of* p *points which are collinear.* [*Cal. Int.* 1928]

Find also the number of triangles formed by them.

If all the n points were such that no three of them are collinear, the number of straight lines formed by joining them would be ${}^nC_2 = \frac{n(n-1)}{2}$.

But p of the points being collinear, ${}^pC_2$, *i.e.*, $\frac{p(p-1)}{2}$ straight lines are lost, and in its place we get only one straight line on which the p points lie.

Hence the required number of straight lines

$$= \frac{n(n-1)}{2} - \frac{p(p-1)}{2} + 1.$$

Again, if no three of the n points were in the same straight line, the number of triangles formed would be

$$^nC_3 = \frac{n(n-1)(n-2)}{1.2.3}.$$

But since p points are in one straight line, the number of triangles lost is $^pC_3 = \frac{p(p-1)(p-2)}{1.2.3}$.

Hence, the required number of triangles

$$= \frac{n(n-1)(n-2)}{6} - \frac{p(p-1)(p-2)}{6}.$$

Example 7. *At an election, where every voter may vote for any number of candidates not greater than the number to be elected there are* 6 *candidates and* 3 *members to be chosen; in how many uays way a man vote?* [*Pun. I. E.* 1893]

A man may vote for any *one* or *two* or *three* of the 6 candidates, and this can be done in $^6C_1$, $^6C_2$ and $^6C_3$ ways.

Hence the total no. of ways a man may vote

$$= {}^6C_1 + {}^6C_2 + {}^6C_3 = 6 + 15 + 20 = 41.$$

155. Restricted Combinations.

To find the number of combinations of n *different things taken* r *at a time in which* (*i*) p *particular things should always occur, and* (*ii*) p *particular things should never occur.*

(*i*) If we set aside the p particular things, there will remain $n-p$ things, and therefore if we select $r-p$ things out of the remaining $n-p$ things in every possible way and combine with them the p particular things, we must get all the required combinations.

Hence the required number in which the p particular things always occur $= {}^{n-p}C_{r-p}$.

(*ii*) Let the p particular things which should never occur in any of the combinations be set aside, then there will remain $n-p$ things, and therefore in order that the p things should never occur, the r things have to be selected out of the $n-p$ things.

Hence the required number $= {}^{n-p}C_r$.

Example 1. *There are* 8 *mangoes and* 22 *other fruits in a basket. In how many ways can* 8 *fruits be selected so as to include at least one mango ?*

There are altogether 30 fruits, and the number of ways in which 8 fruits can be selected out of 30 is ${}^{30}C_8$.

But the number of ways in which 8 fruits can be selected so as not to include any mango is ${}^{22}C_8$.

Hence, the required number of ways in which 8 fruits can be selected so as to include at least one mango

$$= {}^{30}C_8 - {}^{22}C_8$$

$$= \frac{30\cdot29\cdot28\cdot27\cdot26\cdot25\cdot24\cdot23}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8} - \frac{22\cdot21\cdot20\cdot19\cdot18\cdot17\cdot16\cdot15}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7\cdot8}$$

$$= 29.27.13.25.23 - 22.19.3.17.15$$

$$= 5852925 - 319770 = 5533155.$$

Example 2. *If different words be formed with* 5 *only of the letters of the word* Cambridge, *how many of them will contain all the vowels ?*

There are altogether 9 letters of which 3 are vowels. Out of the 6 consonants, 2 consonants can be selected in ${}^6C_2$ ways and therefore if we combine the 3 vowels with each of these selections, we have ${}^6C_2$ selections of the 5 letters containing the 3 vowels. Since each of these combinations of 5 letters produces $|\underline{5}$ words, the required number of words

$$= {}^6C_2 \times |\underline{5} = \tfrac{6}{1}\cdot\tfrac{5}{2} \times |\underline{5} = 1800.$$

Example 3. *In how many ways can* 15 *rupees and* 12 *pice be arranged in a line so that no two pice may occupy consecutive positions ?*

Let us first arrange the 15 rupees in a line. Then, in order that no two pice may occupy consecutive positions, there cannot be more than one pice between any two rupees. Since we have 15 rupees in all, it is evident that there are 14 places possible between the rupees and two other places outside the rupees on either side, thus making altogether 16 places where the pice may be put. Hence we must select 12 positions out of these 16 for the 12 pice, and therefore the required number of ways

$$= {}^{16}C_{12} = {}^{16}C_4 = \frac{16\cdot15\cdot14\cdot13}{1\cdot2\cdot3\cdot4} = 1820.$$

156. Greatest Value of ^nC_r.

We have $$^nC_r=\frac{n(n-1)(n-2)\cdots\cdots(n-r+2)(n-r+1)}{1.\,2.\,3\ldots\ldots(r-1)r}$$

and $$^nC_{r-1}=\frac{n(n-1)(n-2)\ldots\ldots(n-r+2)}{1.\,2.\,3\ldots\ldots(r-1)}.$$

$\therefore$ $$^nC_r={}^nC_{r-1}\times\frac{n-r+1}{r}.$$

Thus, $^nC_r >, =,$ or $< {}^nC_{r-1}$, according as the multiplying factor $\frac{n-r+1}{r} >, =,$ or <1.

Now, as r increases, the numerator of the multiplying factor $\frac{n-r+1}{r}$ diminishes and the denominator increases, and therefore the fraction $\frac{n-r+1}{r}$ diminishes; and as r receives integral values from 1 to n, the multiplying factor $\frac{n-r+1}{r}$ gradually diminishes from n to $\frac{1}{n}$ *i.e.*, from a value >1 to a value <1.

Hence, the terms of the series $^nC_1$, $^nC_2$, $^nC_3,\cdots\cdots{}^nC_n$ continue to increase from the beginning so long as the value of r is such that $\frac{n-r+1}{r}$ is greater than 1, and after this as soon as $\frac{n-r+1}{r}$ becomes less than 1, the terms of the series continue to diminish as r still increases.

Hence, ^nC_r ceases to increase and then attains its greatest value when the multiplying factor $\frac{n-r+1}{r}$ is either just greater than 1, or equal to 1; that is, ^nC_r is greatest when $\frac{n-r+1}{r}$ is either just $>$ or $=1$,

i.e., when $n-r+1$ is either just $>$ or $=r$,
i.e., when $n+1$ is either just $>$ or $=2r$,
i.e., when r is either just $<$ or $=\frac{n+1}{2}$.

Two cases may arise according as n is even or odd.

(i) Let n be *even*; then $n+1$ is odd and $\frac{n+1}{2}$ is a fraction. Since r must be an integer and $\frac{n}{2}$ is the integer just less than $\frac{n+1}{2}$, therefore ${}^nC_r$ is greatest when $r=\frac{n}{2}$; for r, an integer, cannot be equal to $\frac{n+1}{2}$ which is a fraction, and the greatest value that r can have less than $\frac{n+1}{2}$ is $\frac{n}{2}$.

(ii) Let n be *odd*; then $n+1$ is even and $\frac{n+1}{2}$ is an integer. Since r must be an integer either just $<$ or $=\frac{n+1}{2}$, therefore ${}^nC_r$ is greatest when $r=\frac{n+1}{2}$. Since when $r=\frac{n+1}{2}$, the multiplying factor $\frac{n-r+1}{r}=1$, we have ${}^nC_r={}^nC_{r-1}$, *i.e.*, the combinations of n things taken $\frac{n+1}{2}$ at a time is equal to the combinations of n things taken $\frac{n-1}{2}$ at a time. Hence ${}^nC_r$ is also greatest when r is equal to $\frac{n-1}{2}$.

Thus, if n be even, ${}^nC_r$ is greatest when $r=\frac{n}{2}$; and if n be odd, ${}^nC_r$ is greatest when $r=\frac{n+1}{2}$ or $\frac{n-1}{2}$.

Example 1. *Find the greatest number of combinations that can be formed out of* (*i*) 8 *different things, and* (*ii*) 9 *different things.*

(i) ${}^8C_r$ greatest when r is $=\frac{8}{2}=4$.

$\therefore$ the greatest number of combinations $={}^8C_4=\frac{8\cdot7\cdot6\cdot5}{1\cdot2\cdot3\cdot4}=70$.

(ii) ${}^9C_r$ is greatest when $r=\frac{9-1}{2}$ or $\frac{9+1}{2}$, *i.e.*, 4 or 5.

$\therefore$ the greatest number of combinations

$$={}^9C_5={}^9C_4=\frac{9\cdot8\cdot7\cdot6}{1\cdot2\cdot3\cdot4}=126.$$

Example 2. *A Brahmin hospitably disposed wishes to make up as many different parties as he can out of 40 friends, each party consisting of the same number; how many should he invite at a time?* [*Bom. P. E.* 1884]

The question reduces to the finding out of the greatest number of combinations that can be formed out of 40 persons, each combination consisting of the same number of persons. Hence we have to find out the value of r for which ${}^{40}C_r$ is greatest.

$$\therefore \qquad r=\tfrac{1}{2}.40=20.$$

Hence, the Brahmin must invite 20 friends at a time.

EXERCISE 51.

1. Find the values of ${}^{30}C_{26}$ and ${}^{40}C_{35}$.

2. If ${}^{n}C_{7}={}^{n}C_{13}$, find the value of n.

3. If ${}^{n}P_r=720$ and ${}^{n}C_r=120$, find n and r.

4. If ${}^{2n}C_r={}^{2n}C_{r+2}$, find r. [Cal. Int. 1930]

5. Prove that ${}^{n}C_r+2.{}^{n}C_{r-1}+{}^{n}C_{r-2}={}^{n+2}C_r$.

6. Prove that ${}^{n}C_r+3.{}^{n}C_{r-1}+3.{}^{n}C_{r-2}+{}^{n}C_{r-3}={}^{n+3}C_r$.

7. Prove that ${}^{4n}C_{2n}\div{}^{2n}C_n=\dfrac{1.3.5\ \ldots..(4n-1)}{\{1.3.5\ldots\ldots(2n-1)\}^2}$.

8. There are 16 points in a plane, no three of which are in the same straight line. Find the number of straight lines which can be formed by joining them. [Cal. Int. 1909]

9. Four persons are chosen by lot out of ten; in how many ways can this be done, and how often would any person be chosen? [Bom. P. E. 1882]

10. If different words be formed with 6 only of the letters of the word *Question*, how many of them will contain the letters q and n, and how many of them will begin with q and end with n?

11. How many words can be formed with 4 only of the letters of the word *Companies*, so that the letters contained in each word are in alphabetical order?

12. How many different numbers can be formed with 6 of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, (i) arranged in ascending order of magnitude; and (ii) arranged in descending order of magnitude?

13. A basket contains 10 mangoes. Find how many different selections you can make of 3 mangoes so as always to include a particular mango. [Cal. Int. 1921]

14. How many different sums of money can be made up if one of each of the following coins is available : a half-penny , a penny, a sixpence, a shilling, a florin, a pound ?

15. A man possesses 10 coins out of which he has got to take 5 coins. Find in how many of the selections one or both of two given coins will occur. [Cal. Int. 1926]

16. How many combinations can be formed of eight counters marked 1, 2, 3, 4, 5, 6, 7, 8, taking them 4 at a time, there being at least one odd and one even counter in each combination ? [Cal. Int. 1941]

17. Find the number of triangles which can be formed by joining the angular points of a decagon. How many diagonals has it ? [Cal. Int. 1911]

18. There are 10 points in a plane, 4 of which are collinear ; find the number (i) of straight lines, and (ii) of triangles which result from joining them.

19. If there are n points in a plane which are joined in all possible ways by straight lines, and if no two of these straight lines be coincident or parallel and no three pass through the same point, find the number of points of intersection, exclusive of the n given points.

20. In how many ways can m boys and n girls be arranged in a line ($m > n$), if no two girls are together ?

21. A boat's crew consists of 8 men of whom 3 can row only on one side and 2 only on the other. Find the number of ways in which the crew can be selected.

22. A candidate is required to answer six out of ten questions which are divided into two groups each containing five questions, and he is not permitted to attempt more than four from any group. In how many different ways can be make his choice ? [Cal. Int. 1932]

23. A company of 80 men are to be selected from a regiment of 900; find the number of ways in which it can be done so that the same 10 men may be always included. [Pun. I. E. 1890]

24. At an election, there are 5 candidates and 3 members to be elected, and a voter is entitled to vote for any number of candidates not greater than the number to be elected. In how many ways may a voter choose to vote? [Cal. Int. 1935]

25. From 6 gentlemen and 4 ladies, a committee of 5 is to be formed. In how many ways can this be done so as to include at least one lady? [Cal. Int. 1937]

26. A certain council consists of a chairman, two vice-chairmen, and twelve other members. How many different committees consisting of six members can be formed, including always the chairman and only one vice-chairman? [Cal. Int. 1914]

27. A father with 8 children takes three at a time to the Zoological gardens as often as he can without taking the same three children together more than once. How often will he go, and how often will each child go? [Cal. F.A. 1877]

28. A person wishes to make up as many different parties as he can out of 20 persons, each party consisting of the same number; how many should be invited at a time? To how many of these parties should the same person be invited?

III. Harder Permutations and Combinations.

157. Relative Position : Permutations in a ring.

To find the number of ways in which n *persons can form a ring.*

This being a question of *relative position* among the persons, we shall suppose one of the n persons to be in a fixed position and then make all possible arrangements of the remaining $n-1$ persons *with respect to him.* Hence it is clear that the number of ways in which n persons can form a ring is $|\underline{n-1}$, if the arrangement in the clockwise order is considered as different from that in the counter-clockwise order, but if this distinction be not taken into account the number of ways would be $\frac{1}{2}\,|\underline{n-1}$.

But if the n persons, instead of forming a ring, be required to be seated at a round table, this can be done in $|\underline{n}$ ways, for in this case the different arrangements of the positions of the persons are with respect to the table and not with respect to each other.

Similarly, if n different beads are to be arranged to form a necklace, the number of different orders in which they can be put is $\frac{1}{2}\,|\underline{n-1}$, for if the necklace containing the beads in any order be turned over, the arrangement of the beads in the clockwise order at any stage becomes identical with the arrangement in the counter-clockwise order.

158. Permutations and Combinations formed from different sets of things.

If there are several sets of things the first set containing m *different things, the second set* n *different things, the third set* p *different things and so on, then the number of combinations formed by taking* r *at a time from the first set,* s *from the second,* t *from the third, etc. is* ${}^mC_r \times {}^nC_s \times {}^pC_t \times \cdots$; *and the corresponding number of permutations that can be formed with them is*

$$ {}^mC_r \times {}^nC_s \times {}^pC_t \times \cdots |\underline{r+s+t+\cdots} $$

Since r things may be selected out of the m things in ${}^mC_r$ ways, and since corresponding to each of the ${}^mC_r$ ways of selecting the r things, there are ${}^nC_s$ ways of selecting s things out of the n things the total number of combinations that can be formed with r and s things is evidently ${}^mC_r \times {}^nC_s$.

Again, since t things may be selected out of the p things in ${}^pC_t$ ways, and since corresponding to each of the ${}^mC_r \times {}^nC_s$ ways of selecting the $r+s$ things, there are ${}^pC_t$ ways of selecting the t things, the total number of combinations that can be formed with the r, s and p things is ${}^mC_r \times {}^nC_s \times {}^pC_t$.

Similarly, continuing this process, whatever be the number of sets of things, the number of combinations formed in the same way will be ${}^mC_r \times {}^nC_s \times {}^pC_t \times \cdots\cdots$

Since each of the combinations contains $r+s+t+\cdots$ things and therefore produces $|\underline{r+s+t+\cdots}$ permutations when the things contained are arranged among themselves, the required number of permutations is evidently

$$ {}^mC_r \times {}^nC_s \times {}^pC_t \times \cdots \times |\underline{r+s+t+\cdots}. $$

Cor. If there are k sets of things containing $m, n, p \cdots$ things respectively, then the number of combinations formed by taking *one* out of each set $= mnp \cdots$; and the corresponding number of permutations $= mnp \ldots \times |\underline{k}$.

159. Division into Groups.

(*i*) *To find the number of ways in which* m+n *things can be divided into two groups of* m *and* n *things respectively.*

Since after selecting the m things for the first group from the $m+n$ things which can be done in ${}^{m+n}C_m$ ways, there always remain n things to form the second group, it is clear that the number of ways in which the $m+n$ things can be divided into two groups of m and n things respectively, must be the same as the number of ways in which m things can be selected from the $m+n$ things. Hence the required number of ways

$$= {}^{m+n}C_m = \frac{\lfloor m+n}{\lfloor m \; \lfloor m+n-m} = \frac{\lfloor m+n}{\lfloor m \; \lfloor n}.$$

If $m = n$, the two groups are equal and therefore the number of ways in which $2m$ things can be distributed equally to two persons is $\frac{\lfloor 2m}{\{\lfloor m\}^2}$. But the number of ways in which $2m$ things can be divided into two equal groups is $\frac{\lfloor 2m}{\{\lfloor m\}^2 \lfloor 2}$ or $\frac{1}{2} \cdot \frac{\lfloor 2m}{\{\lfloor m\}^2}$, for in this case the two different orders in which the two groups can be arranged are considered as equivalent.

(*ii*) *To find the number of ways in which* m+n+p *things can be divided into three groups of* m, n *and* p *things respectively.*

After selecting the m things for the first group from $m+n+p$ things which can be done in ${}^{m+n+p}C_m$ ways, and then selecting the n things for the second group from the remaining $n+p$ things in ${}^{n+p}C_n$ ways, there will remain p things which will evidently form the third group.

Hence the required number of ways

$$= {}^{m+n+p}C_m \times {}^{n+p}C_n = \frac{\lfloor m+n+p}{\lfloor m \; \lfloor n+p} \times \frac{\lfloor n+p}{\lfloor n \; \lfloor p} = \frac{\lfloor m+n+p}{\lfloor m \; \lfloor n \; \lfloor p}.$$

If $m = n = p$, the three groups are equal and therefore the number of ways in which $3m$ things can be distributed equally to three persons is $\frac{\lfloor 3m}{\{\lfloor m\}^3}$. But the number of ways in which $3m$ things can be divided into 3 equal groups

is $\dfrac{\lfloor 3m}{\{\lfloor m\}^3 \lfloor 3}$ or $\dfrac{1}{6}\cdot\dfrac{\lfloor 3m}{\{\lfloor m\}^3}$, for in this case the three groups can be arranged in $\lfloor 3$ different orders and they are all considered as equivalent.

160. Total number of Combinations.

To find the total number of combinations of n *different things taken any number at a time, i.e., to find the value of* $^nC_1 + {^nC_2} + {^nC_3} + \cdots + {^nC_n}$.

In making a selection out of the n things, each thing may be disposed of in 2 ways, namely, it may be either selected or not selected; and each way of disposing of any one thing may be combined with each way of disposing of any one of the remaining $n-1$ things. Hence the total number of selections is $2 \times 2 \times 2 \times \cdots$ to n factors $= 2^n$. But this includes the case in which none of the n things are selected, which is inadmissible.

Hence the required number of combinations $= 2^n - 1$.

Example 1. *In how many ways can* 7 *persons form a ring?*

Supposing one of the 7 persons to be in a fixed position and then making all possible arrangements of the remaining 6 persons, it is evident that the number of ways in which 7 persons can form a ring is $\lfloor 6$, *i.e.*, 720.

But if the distinction between clockwise and counter-clockwise order be not considered as different, the number of ways is $\frac{1}{2}\lfloor 6$, *i.e.*, 360.

Example 2. *In how many ways can* 6 *persons be seated at a round table?*

If we suppose the different arrangements of the positions of the persons to be with respect to the seats connected with the table, the no. of arrangements $= \lfloor 6$.

But if the arrangements of the persons are supposed to be with respect to each other, the no. of arrangements $= \lfloor 5$, for, in this case one of the 6 persons is to be fixed in position and the remaining 5 persons are to be arranged in all possible ways.

In the latter case, if the clockwise and counter-clockwise order be not regarded as different, the no. of ways $= \frac{1}{2}\lfloor 5$.

Example 3. *In how many ways can* 5 *gentlemen and* 5 *ladies be seated at a round table differently with respect to each other so that no two ladies may be together.*

The number of ways in which the 5 gentlemen may be seated $= \lfloor 4 = 24$.

In order that no two ladies may be together, they have to be seated, one between two gentlemen and they may be arranged among themselves in $\lfloor 5$ ways. Hence the total number of ways in which the gentlemen and the ladies may be seated $= \lfloor 4 \times \lfloor 5 = 2880$.

This will be the result if the clockwise and counter-clockwise order are regarded as different, but if they are considered as equivalent, the number of ways would be $\frac{1}{2} \times \lfloor 4 \times \lfloor 5 = 1440$.

Example 4. *From 10 gentlemen and 7 ladies how many different parties can be formed, each consisting of 3 gentlemen and 2 ladies ?*

The gentlemen can be selected in ${}^{10}C_3 = \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} = 120$ ways, and the ladies can be selected in ${}^{7}C_2 = \frac{7 \cdot 6}{1 \cdot 2} = 21$ ways.

Hence, the required no. of different parties $= 120 \times 21 = 2520$.

Example 5. *From 21 consonants and 5 vowels, how many words of 5 letters can be formed, each containing 3 consonants and 2 vowels ?*

Three consonants can be selected in ${}^{21}C_3 = \frac{21 \cdot 20 \cdot 19}{1 \cdot 2 \cdot 3} = 1330$ ways, and two vowels can be selected in ${}^{5}C_2 = \frac{5 \cdot 4}{1 \cdot 2} = 10$ ways.

Hence, the number of combinations each containing 3 consonants and 2 vowels $= 1330 \times 10 = 13300$.

Since each combination of the 5 letters produces $\lfloor 5$ different words, the required number of words that can be formed $= 13300 \times \lfloor 5 = 1596000$.

Example 6. *In how many ways can 12 things be divided into three groups of 3, 4 and 5 things respectively ?*

The number of ways in which 3 things can be selected, for the first group, from the 12 things $= {}^{12}C_3$; the number of ways in which 4 things can be selected for the second group from the remaining 9 things $= {}^{9}C_4$; and after selecting for the first and second groups, there will remain 5 things which will evidently form the third group.

Hence, the required number of ways

$$= {}^{12}C_3 \times {}^{9}C_4 = \frac{12 \cdot 11 \cdot 10}{1 \cdot 2 \cdot 3} \times \frac{9 \cdot 8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3 \cdot 4} = 220 \times 126 = 27720.$$

Example 7. *In how many ways can 52 cards be divided equally among 4 players ? In how many ways can 52 cards be divided into 4 packets of 13 each ?*

(i) Let A, B, C, D be the four persons.

Then 13 cards can be selected for A from 52 cards in ${}^{52}C_{13}$ ways; after selecting for A, 13 cards can be selected for B from the remaining 39 cards in ${}^{39}C_{13}$ ways; then after selecting for A and B, 13 cards can be selected for C from the remaining 26 cards in ${}^{26}C_{13}$ ways; and lastly after selecting for A, B and C, the remaining 13 cards can be given to D. Hence the total number of ways

$$={}^{52}C_{13}\times{}^{39}C_{13}\times{}^{26}C_{13}$$

$$=\frac{\lfloor 52}{\lfloor 13\lfloor 39}\times\frac{\lfloor 39}{\lfloor 13\lfloor 26}\times\frac{\lfloor 26}{\lfloor 13\lfloor 13}=\frac{\lfloor 52}{\{\lfloor 13\}^4}.$$

(ii) The number of ways in which 52 cards can be divided into 4 packets of 13 each is $\frac{\lfloor 52}{\lfloor 4\{\lfloor 13\}^4}$, for in this case the 4 packets can be arranged in $\lfloor 4$ different orders, and these $\lfloor 4$ different orders are regarded as equivalent.

Example 8. *How many different sums of money can be formed with a rupee, an eight-anna piece, a four-anna piece, a two-anna piece and a one-anna piece?*

Since there are altogether 5 different coins, the number of different sums that can be formed

$$={}^5C_1+{}^5C_2+{}^5C_3+{}^5C_4+{}^5C_5=2^5-1=31.$$

Example 9. *How many different products can be formed with n different letters?*

Since to form a product at least two of the n letters must be taken and a single letter is not a product, the required number of the products

$$={}^nC_2+{}^nC_3+{}^nC_4+\cdots\cdots+{}^nC_n=(2^n-1)-{}^nC_1=2^n-n-1.$$

EXERCISE 52.

1. Find in how many ways 10 children can sit in a merry-go-round relatively to one another. [Cal. Int. 1927]

2. In how many ways can 10 persons be seated at a round table?

3. In how many ways can 7 beads of different colours be strung on a necklace?

4. In how many ways can 7 gentlemen and 7 ladies be seated at a round table with respect to each other, so that no two ladies may be together ?

5. In how many ways can m men and n women $(m>n)$ be arranged in a ring, so that no two women may stand together ?

6. Out of 9 Swarajists and 5 Ministerialists, how many different committees can be formed, each consisting of 6 Swarajists and 3 Ministerialists ? [Cal. Int. 1931]

7. In how many ways can a committee of 3 ladies and 4 gentleman be appointed from a meeting consisting of 18 ladies and 12 gentlemen ? [Cal. Int. 1923]

8. How many words of 2 vowels and 3 consonants can be formed from an alphabet of 5 vowels and 17 consonants, the letters of the words being all different ? [Cal. Int. 1939]

9. Find the number of words which can be formed with two different consonants and one vowel, out of 7 different consonants and 3 different vowels, the vowel to lie between the two consonants. [Cal. Int. 1922]

10. A cricket team consisting of 11 players is to be selected from two groups consisting of 6 and 8 players respectively. In how many ways can the selection be made on the supposition that the group of 6 shall contribute no fewer than 4 players ? [Cal. Int. 1938]

11. In how many ways can 12 different things be divided equally among 3 persons ?

12. In how many ways can 12 different books be divided into 3 sets, each containing 4 books ?

13. In how many ways can mn different things be divided among n persons so that each may have m of them ?

14. In how many ways can mn different things be divided into n sets each of m things ?

15. Find the number of combinations of $2n$ things of which n are alike, taken n at a time.

16. A gentleman invites a party of $m+n$ friends to dinner and places m at one round table and n at another ; find the number of ways in which he can arrange them *among themselves.* [Bom. P. E. 1893]

17. How many different weights can be formed with a pound, a half-pound, a four-ounce, a two-ounce and a one-ounce weight?

18. In how many different ways can I invite one or more of 8 friends to dinner?

19. There are six papers set in an examination and a minimum is to be secured in each paper for a pass. In how many ways can a student fail?

20. In how many ways can a pack of 52 cards be distributed equally among four persons so that (*a*) each may have ace, king, queen and knave of the *same suit*; and (*b*) each may have ace, king, queen and knave *all of different suits*?

[Mad. F. A. 1878]

161. Permutations of things not all different.

To find the number of permutations of n *things taken all together, when the things are not all different.*

Let the n things be represented by n letters, and suppose that p of them are a's, q of them are b's, r of them are c's, and the rest are all different.

Let x be the required number of permutations.

If the p a's had been all different, they could have been arranged in $\lfloor p$ ways among themselves. Similarly, if the q b's and r c's had been all different, they could have been arranged in $\lfloor q$ and in $\lfloor r$ ways respectively among themselves. Hence, if the n things had been all different, the total number of possible permutations would have been $x \times \lfloor p \times \lfloor q \times \lfloor r$. But this number is evidently $\lfloor n$.

Hence, $\quad x \times \lfloor p \times \lfloor q \times \lfloor r = \lfloor n$

$$\therefore \qquad x = \frac{\lfloor n}{\lfloor p \lfloor q \lfloor r},$$

which is the required number of permutations.

162. Permutations of things which may be repeated.

To find the total number of permutations of n *different things taken* r *at a time, when each thing may be repeated up to* r *times in any arrangement.*

Let the n things be represented by the n letters a, b, c, d, ... p. Then the required number of permutations will be the same as the number of ways in which r places can be filled by using any one of these letters in each place.

The first place may be filled up in n ways for we may use any one of the n letters. After the first place is filled up in any one way the second place may be filled up in n ways for we may repeat the same letter again. Therefore the number of ways in which the first two places may be filled up is $n \times n$, *i.e.*, n^2.

After the first two places are filled up in any one of the n^2 ways, the third place may also be fillled up in n ways, and therefore the first three places may be filled up in $n^2 \times n$ ways, *i.e.*, n^3 ways.

Since at any stage the index of n is always the same as the number of places filled up, by continuing the above process it is evident that the number of ways in which r places may be filled up is equal to n^r.

Example 1. *Find the number of permutations of the letters of the word* Calcutta, *taken all together.* [*Cal. Int.* 1916]

How many of these begin and end with c ?

We have 8 letters in the word *Calcutta* of which 2 are c, 2 are a, 2 are t and the remaining 2 are different letters.

Hence, the required number of words

$$=\frac{\lfloor 8}{\lfloor 2 \lfloor 2 \lfloor 2}=1.2.3.4.5.6.7=5040.$$

The number of words which begin and end with c is the same as the number of permutations of the remaining letters a, l, u, t, t, a taken all together, and therefore

$$=\frac{\lfloor 6}{\lfloor 2 \lfloor 2}=180.$$

Example 2. *How many numbers can be formed with the digits* 2, 3, 3, 4, 4, 4, 5, 5, 5, *so that the odd digits always occupy the odd places ?*

Since we have altogether 9 digits, there are 5 odd and 4 even places.

The odd digits 3, 3, 5, 5, 5 can be arranged in the 5 odd places in $\frac{\lfloor 5}{\lfloor 2 \lfloor 3}$ or 10 ways.

The even digits 2, 4, 4, 4 can be arranged in the 4 even places in $\frac{\lfloor 4}{\lfloor 3}$, or 4 ways.

Since each of the ways of arranging the odd digits can be associated with each of the ways of arranging the even digits, the required number $=10 \times 4=40$.

Example 3. *How many different numbers, each consisting of 4 digits can be formed with the digits* 1, 2, 3, 4, 5, 6, 7, *when each digit may occur any number of times ?*

This is the same as the number of permutations of 7 things taken 4 at a time when each thing may occur any ber of times. Hence the required number $=7^4=2401$.

mple 4. *In how many ways can* 5 *prizes be given away to* 4 *boys when each is eligible for any number of prizes ?*

Since there are 4 boys, any one of the prizes can be given in 4 ways ; then any one of the remaining prizes can also be given in 4 ways, for the boy who has already received the first prize may also obtain the second one. Hence, the two prizes can be given in 4^2 ways. Similarly, three prizes can be given in 4^3 ways, and so on. Thus, the 5 prizes can be given in 4^5 ways, or 1024 ways.

EXERCISE 53.

1. How many different permutations can be made out of the letters of the following words taken all together :

(i) *Critical* ; (ii) *Proportion* ; (iii) *Assassination.*

2. How many different words can be formed out of the letters of the word *Constantinople* ? In how many of these will the 3 *n*'s be consecutive letters ? [Cal. F. A. 1876]

3. How many different numbers each of 6 digits can be expressed by means of the digits of the number 121202 ? [Mad. F. A. 1896]

4. How many even numbers each of the 7 digits can be formed with the digits 4, 5, 5, 6, 7, 7, 7 ?

5. How many different words can be made out of the letters which form the word *Allahabad* ? In how many of these will the vowels occupy the even places ? [All. I.E. 1891]

6. How many different signals can be made by using 9 flags all together, of which 2 are red, 3 black, and 4 white ?

7. A telegraph has 5 arms and each arm is capable of 4 distinct positions, including the position of rest. What is the total number of signals that can be made ? [Cal. Int. 1913]

8. In how many ways can the letters of the word *Arrange* be arranged ? How many arrangements can be made (*a*) if the two *r*'s are not allowed to come together ; and (*b*) if neither the two *r*'s nor the two *a*'s are allowed to come together ? [Mad. F. A. 1888]

9. In how many ways can the letters of the word *Consonant* be arranged :

(i) without changing the order of the vowels ;

(ii) without changing the place of any vowel ;

(iii) without changing the relative order of vowels and consonants ;

(iv) without changing the order of the vowels and also the order of the consonants ?

10. In how many ways can the letters of the word *Civilization* be re-arranged without changing the relative order of vowels and consonants ?

11. A library has 5 copies of one book, 4 copies of each of two books, 6 copies of each of three books, and single copies of eight books. In how many ways can all the books be arranged ? [Cal. Int. 1934]

12. How many numbers can be formed with the digits of the number 47263215 without changing the order of the odd digits ?

13. In the ordinary system of notation, how many numbers are there which consist of 4 digits ?

14. A letter-lock consists of 5 rings each marked with 6 different letters ; how many unsuccessful attempts can be made to open the lock ?

15. In how many ways can 8 things be divided between 2 men ?

16. There are 5 candidates for a post and 7 electors. In how many ways can the votes be given ?

17. In how many ways can 3 things be given to 5 persons ?

18. There are 5 letters to be sent and three post-offices within reach. In how many ways can the letters be posted ?

19. In how many ways can 6 letters be placed in 6 envelopes, one in each, if two of the letters are too large for one of the envelopes ?

20. How many numbers of 5 digits can be formed with 0, 1, 2, 3, 4, 5, 6, 7, if each of these digits may be repeated ? How many of these will be greater than 50000 ?

21. There are m men and n monkeys, n being greater than m. If a man may have any number of monkeys, in how many ways may every monkey have a master ? [Bom. P. E. 1891]

22. Show that the total number of permutations of n different things taken not more than r at a time, when any of the n letters may be repeated, is $\frac{n(n^r-1)}{n-1}$.

163. Combinations of things not all different.

To find the total number of combinations of n *things, taking any number from* 1 *to* n *at a time, when things are not all different.*

Let the n things contain p things of one kind, q things of a second kind, r things of a third kind, and so on. In making a selection out of the n things, the p like things of one kind may be disposed of in $p+1$ ways, for we may either select $1, 2, 3, \cdots p$ things at a time or reject all of them. Similarly, the q like things may be disposed of in $q+1$ ways, the r like things in $r+1$ ways, and so on.

Since each of the $p+1$ ways of disposing of the p like things may be combined with each of the $q+1$ ways of disposing of the q like things, the number of ways in which the p and q things may be disposed of is $(p+1)\,q+1)$. Again, since each of the $(p+1)(q+1)$ ways of disposing of the $p+q$ things may be combined with each of the $r+1$ ways of disposing of the r things, the number of ways in which the p, q and r things may be disposed of is $(p+1)(q+1)(r+1)$; and so on. Hence the total number of ways in which all the things may be disposed of is

$$(p+1)(q+1)(r+1)\cdots\cdots$$

But this includes the case when all the things are rejected, which case is inadmissible.

Hence the required total number of combinations

$$=(p+1)(q+1)(r+1)\cdots \quad -1.$$

Cor. If $p=q=r=\cdots=1$, that is, if all the things are different, the total number of combinations of n things taken $1, 2, 3, \ldots n$ at a time

$$=(1+1)(1+1)(1+1)\cdots\cdots \quad -1$$
$$=2.\,2.\,2 \cdots\cdots n \text{ factors} \quad -1$$
$$=2^n-1.$$

[See Art. 160]

Example 1. *Find the number of factors of* 21600.

Since $21600 = 2^5 \times 3^3 \times 5^2$, *i.e.*, the product of 5 2's, 3 3's and 2 5's, we have the total no. of factors

$$=(5+1)(3+1)(2+1)-1=6\times4\times3-1=71.$$

But this includes the number 21600 itself, when the product of all the factors is taken. Hence rejecting this, we have the required no. of factors $=71-1=70$.

Example 2. *Show that the number of all possible selections of one or more questions from eight given questions, each question having an alternative, is* 3^8-1. [*Cal. Int.* 1928]

Since each question has an alternative, we may deal with each question in 3 ways, namely, we may select either the question itself or its alternative, or reject both of them, and therefore we may deal with the eight questions in 3^8 ways. But this includes the case when all the questions with their alternatives are rejected. Omitting this case, we have the required number of selections $=3^8-1$.

Exercise 54.

1. Find the number of factors of 9800.

2. There are 5 kinds of books, and 3 of each kind. In how many ways can a selection be made from them ?

3. There are 3 rupees, 4 eight-anna pieces, 5 four-anna pieces, 6 two-anna pieces, and 7 one-anna pieces. How many different sums of money can be formed with them ?

4. Find the number of ways in which 1440 can be expressed as the product of two integral factors.

5. There are 8 Hindus and 10 Mahomedans. How many different parties can be formed, taking at least one of each sect ?

6. There are 3 oranges, 4 pears and 5 apples. In how many ways can a selection be made from them, if at least one of each kind is to be taken ?

7. If there be m sorts of things and n things of each sort, prove that the number of ways in which a selection can be made from them is $(n+1)^m-1$.

8. If of $p+q+r$ things, p be alike, q be alike, and the rest be all unlike, prove that the total number of combinations is $(p+1)(q+1)2^r-1$.

164. Permutations and Combinations of n things taken r at a time when the things are not all different.

The general formulæ in this case are very intricate, and therefore we shall illustrate the method by solving a few examples.

Example 1. *Find the number of combinations of the letters of the word* Proportion *taken* 4 *at a time.*

There are altogether 10 letters namely, $p, p, r, r, o, o, o, t, i, n$, and of these the different letters are p, r, o, t, i, n, 6 in number.

In forming combinations of these letters taken 4 at a time, we may classify them in the following manner :

(i) All the four letters different.
(ii) Two letters different and other two alike.
(iii) One letter different and the other three alike.
(iv) Two pairs of like letters.

(i) Since there are 6 different letters p, r, o, t, i, n, the number of combinations taking 4 at a time $= {}^6C_4 = 15$.

(ii) Combinations of this class are formed by combining two p's with any two of the 5 letters r, o, t, i, n, and this can be done in ${}^5C_2$, *i.e.*, 10 ways ; combining two r's with any two of the 5 letters p, o, t, i, n, and this can be done in ${}^5C_2$, *i.e.*, 10 ways ; and combining two o's with any two of the 5 letters p, r, t, i, n, and this can be done in ${}^5C_2$, *i.e.*, 10 ways. Thus the total no. of combinations of this class $= 10 + 10 + 10 = 30$.

(iii) Combinations of this class are formed by combining the three o's with any one of the 5 letters, and this can be done in ${}^5C_1$, *i. e.*, 5 ways.

(iv) Combinations of this class are formed by taking any two pairs of like letters, two p's, two r's and two o's, and this can be done in ${}^3C_2$, *i.e.*, 3 ways.

Hence the required no. of combinations $= 15 + 30 + 5 + 3 = 53$.

Example 2. *Find the number of permutations of the letters of the word* Proportion *taken* 4 *at a time.*

As in Example 1 above, classifying the combinations under the four heads, we proceed as follows :

(i) In this case all the letters are different, and each combination of 4 different letters produces $\lfloor 4$ permutations. Since there are 15 combinations of this class, the number of permutations $= 15 \times \lfloor 4 = 360$.

(ii) In this case each combination contains 4 letters of which two are alike, and therefore produces $\frac{\lfloor 4}{\lfloor 2}$, *i.e.*, 12 permutations. Since there are 30 such combinations, the number of permutations $=30 \times 12=360$.

(iii) In this case each combination contains 4 letters of which 3 are alike, and therefore produces $\frac{\lfloor 4}{\lfloor 3}$, *i.e.*, 4 permutations. Since there are 5 such combinations, the number of permutations $=5 \times 4=20$.

(iv) In this case each combination of 4 letters contains two pairs of like letters, and therefore produces $\frac{\lfloor 4}{\lfloor 2 \lfloor 2}$, *i.e.*, 6 permutations. Since there are 3 such combinations, the number of permutations $=3 \times 6=18$.

Hence the total no. of permutations

$$=360+360+20+18=758.$$

EXERCISE 55.

1. Find the number of combinations of the letters of the word *Alliteration* taken 4 at a time. Find also the corresponding number of permutations. [Cal. F. A. 1889]

2. Find the number of combinations of the letters of the word *Accommodation* taken 4 at a time.

3. How many different words can be formed with the letters composing the word *Examination* (i) taken 3 at a time and (ii) taken 4 at a time ?

4. Find the number of different arrangements that can be made from the letters a, a, a, b, b, c, d, e, taken 4 at a time.

5. How many different numbers can be formed with the digits 2, 2, 2, 3, 3, 4, 4, 5, 6, taken 6 at a time ?

6. Find how many numbers greater than 1000 can be formed with the digits 112340, taken 4 at a time.

7. Find the number (i) of combinations, and (ii) of permutations that can be made from the letters of the word *Expression* taken 4 at a time.

8. Find the number of combinations of the letters of the word *Consonant* taken 5 at a time.

9. How many words, each consisting of two vowels and two consonants, can be made out of the letters of the word *Devastation*? In how many of them will the two *t*'s be together? [Cal. F. A. 1897]

10. Show that the number of permutations 4 at a time which can be made of n groups of things, of which each group consists of three things like one another but unlike the rest, is $n(n-1)(n^2+n+1)$. [Bom. P. E. 1902]

165. We proceed to work out below some examples in Permutations and Combinations of miscellaneous types.

Example 1. *Find the sum of all the numbers that can be formed with all the digits* 1, 2, 3, 4, 5, 6 *in the scale of* 10.

It is evident that using all the six digits, ${}^6P_6$, *i.e.*, 720 numbers can be formed. If we suppose all these 720 numbers written down one below another, the digit 1 occurs in the unit's place of ${}^5P_5$, *i.e.*, 120 numbers; and each of the other 5 digits also occurs similarly in the unit's place of 120 numbers in each case. Hence the sum of all the digits in the unit's place of the 720 numbers

$$=1\times120+2\times120+3\times120+4\times120+5\times120+6\times120$$
$$=(1+2+3+4+5+6)\times120=21\times120.$$

Similarly, the sum of all the digits in the second column from the right is also $=21\times120$; and for the same reason, the sum of all the digits in each of the other columns is also $=21\times120$.

Hence the sum of all the numbers required

$$=21\times120+21\times120\times10+21\times120\times10^2$$
$$+21\times120\times10^3+21\times120\times10^4+21\times120\times10^5$$
$$=21\times120\times(1+10+10^2+10^3+10^4+10^5)$$
$$=21\times120\times111111=279999720.$$

NOTE. Similarly, it can be shown that if there are r different digits $a, b, c, d, \ldots$ and none of them is 0, the sum of all the numbers that can be formed, using all the digits,

$$=(a+b+c+d+\ldots\ldots)\times\underline{|r-1}\times(1+10+10^2+\ldots\ldots+10^{r-1})$$

Example 2. *Find the sum of all the numbers that can be formed by the permutations of the digits* 1, 1, 2, 2, 2, 3, 4, 5.

If we suppose all the numbers, thus formed, written down one below another, the digit **1** occurs in the unit's place of $\frac{\underline{|7}}{\underline{|3}}$, *i.e.*, 840 numbers, the digit 2 in the unit's place of $\frac{\underline{|7}}{\underline{|2}\,\underline{|2}}$, *i.e.*, 1260 numbers, the digit 3 in the unit's place of $\frac{\underline{|7}}{\underline{|2}\,\underline{|3}}$ *i.e.*, 420 numbers, the digit 4 in the unit's place of $\frac{\underline{|7}}{\underline{|2}\,\underline{|3}}$, *i.e.*, 420 numbers, and the digit 5 in the unit's place of $\frac{\underline{|7}}{\underline{|2}\,\underline{|3}}$, *i.e.*, 420 numbers. Hence the sum of all the digits in the unit's place of all the numbers

$$=840\times1+1260\times2+420\times3+420\times4+420\times5$$
$$=840+2520+1260+1680+2100$$
$$=8400.$$

Similarly, the sum of all the digits in each of the other 7 columns of all the numbers is also $=8400$.

Hence the sum of all the numbers required

$$=8400\times(1+10+10^2+10^3+10^4+10^5+10^6+10^7)$$
$$=8400\times11111111=93333332400.$$

Example 3. *There are* 3n *things, of which* 2n *are alike and the rest all different; find the number of combinations of them taken* 2n *at a time.*

Since there are $2n$ things alike and n things all different, we may take all the $2n$ like things and this can be done in only 1 way.

Then we may take any number of things from 1 to n out of the n different things and the rest out of the $2n$ like things to make the total number $2n$, and they can be done in ${}^nC_1$, ${}^nC_2$, ${}^nC_3$, $\cdots {}^nC_n$ ways respectively.

Hence, the total number of combinations required

$$=1+{}^nC_1+{}^nC_2+{}^nC_3+\cdots+{}^nC_n=2^n. \qquad \text{[Art. 160]}$$

Example 4. *Prove that*

$$^{m+n}C_r = {}^mC_r + {}^mC_{r-1}\,{}^nC_1 + {}^mC_{r-2}\,{}^nC_2 + \cdots + {}^mC_1\,{}^nC_{r-1} + {}^nC_r,$$

where r *is not greater than either of* m *and* n.

Let $a, b, c, \ldots p$ be m things and $a', b', c', \cdots q'$ be n things.

Then $^{m+n}C_r$ = the total number of combinations of the $m+n$ things taken r at a time. Now it is easy to see that all these combinations may be formed in the following way :

(i) All the r things may be selected from the first group ; and this can be done in ^mC_r ways ;

(ii) $r-1$ things may be selected from the first group, and 1 thing from the second ; and this can be done in $^mC_{r-1} \times {}^nC_1$ ways ;

(iii) $r-2$ things may be selected from the first group and 2 things from the second ; and this can be done in $^mC_{r-2} \times {}^nC_2$ ways ; and so on.

Hence, proceeding in this way, we have

$$^{m+n}C_r = {}^mC_r + {}^mC_{r-1}\,{}^nC_1 + {}^mC_{r-2}\,{}^nC_2 + \cdots + {}^mC_1\,{}^nC_{r-1} + {}^nC_r.$$

Example 5. *There are two groups of things, one group containing* m *different things and the other* n *different things. Find the number of ways in which the things in the two groups can be exchanged without altering the number in each group.*

Since there are altogether $m+n$ different things, it is easy to see that the required number of ways is the same as the number of ways in which the $m+n$ things can be divided into two groups, one containing m things and the other n things ; and this is done in $^{m+n}C_m$ ways. But this includes the case of the original division of the things into two groups.

Hence, the required no. of ways $= {}^{m+n}C_m - 1 = \dfrac{\lfloor m+n}{\lfloor m \; \lfloor n} - 1.$

EXERCISE 56.

1. Find the sum of all the numbers which can be formed with all the digits 1, 2, 3, 4, 5, in the scale of 10.

2. Find the sum of all the numbers which can be formed with the digits 0, 1, 2, 3, 4, in the denary scale.

3. Find the sum of all the numbers that can be formed with the digits 2, 3, 3, 4, 4, 4, in the decimal scale.

4. A has 8 books, B has 5 books and C has 7 books; in how many ways can they mutually exchange their books without altering the number of books each of them had at first?

5. In how many ways can a pack of 52 cards be arranged so that (i) all the cards of one kind may be consecutive; and (ii) all the cards of one kind may be consecutive and in proper order?

6. Find how many different words can be formed with 5 given letters. of which 3 are consonants and 2 are vowels, no two consonants being juxtaposed in any word.

[Cal. Int. 1929]

7. How many different collections of letters can be made by taking at least one letter from each of the words *Change*, *Number* and *Factor*?

8. Four letters are written and 4 envelopes directed. Find the number of ways in which the letters may be put into the envelopes so that all the letters will be wrongly directed.

9. Show that the number of different algebraical quantities that can be formed by combining the m quantities $a, b, c, d, \ldots\ldots$ by addition or subtraction, using the signs $+$ and $-$, is 2^m.

10. A gentleman invites a party of 10 friends to dinner, and places half of them on one side of a long table and half on the other side; if 3 particular men desire to sit on one side, and 2 on the other, find the number of ways in which he can arrange the guests.

11. A boat's crew consists of $2n$ men, p of whom can only row on one side and q only on the other. Find the number of ways in which the crew can be arranged.

12. Twenty-two men arrange to play a cricket match. If two of the men are brothers, find out the number of ways in which the teams can be made up so that the brothers do not play on the same side.

13. Find the greatest number of points of intersection of (i) 12 straight lines with each other; and (ii) 12 circles with each other.

14. There are 10 letters of which some are alike and the rest are all different; if 5040 words can be formed with them taking all together, how many letters are alike?

15. A cricket party on tour consists of 19 men, of whom 12 are primarily batsmen, 5 primarily bowlers, 2 primarily wicket-keepers; in how many ways can an eleven consisting of 7 batsmen, 3 bowlers, and 1 wicket-keeper be selected?

16. Find in how many ways groups of 8 persons, 4 ladies and 4 gentlemen, can be formed out of 8 ladies and 9 gentlemen, subject to the condition that two particular gentlemen are not simultaneously to be in the same group with a particular lady. [Mad. F. A. 1885]

17. There are n points in space, no three of which are in the same straight line and no four in the same plane, with the exception of p points which are all in one plane; find the number of different planes which can be formed by them.

18. The number of combinations of n letters, taken 5 together, in which a, b, c, occur is 21. Find the number of combinations of them, taken 6 together, in which a, b, c, d occur.

19. A bag contains n sovereigns and n shillings. Prove that the number of ways in which they can be drawn out in succession, one at a time, is $\dfrac{1.3.5\cdots(2n-1)}{\lfloor n}.2^n$.

20. If m points in one straight line are joined to n points in another straight line, how many points of intersection are there besides the original m and n points?

21. If P_r denotes the number of permutations of n different things taken r at a time, show that

$$P_1+\frac{P_2}{\lfloor 2}+\frac{P_3}{\lfloor 3}+\cdots+\frac{P_n}{\lfloor n}=2^n-1. \qquad \text{[Mad. F.A. 1880]}$$

22. If X_r denote the number of permutations of X different things taken r together, show that

$$(M+N)_r=M_r+r.M_{r-1}.N_1+\frac{r(r-1)}{1.2}.M_{r-2}N_2+\cdots+rM_1.N_{r-1}+N_r.$$

[Mad. F.A. 1878]

23. If c_r denote the combinations of n different things taken r at a time, show that

$$c_1{}^2+c_2{}^2+c_3{}^2+\cdots+c_n{}^2=\frac{\lfloor 2n}{\{\lfloor n\}^2}-1.$$

24. There are $n-1$ sets containing $2a$, $3a$,$\cdots\cdots na$ things respectively; show that the number of combinations, which can be formed by taking a out of the first set, $2a$ out of the second set, and so on, for each combination, is $\dfrac{\lfloor na}{(\lfloor a)^n}$.

CHAPTER XV

BINOMIAL THEOREM

166. The *Binomial Theorem* is a general algebraical formula, by means of which any power or root of a binomial expression, *i.e.*, an expression containing two terms, can be expressed in the form of a series. This theorem was first enunciated by Sir Isaac Newton. We shall first take up the case when the index (or exponent) of the power is a positive integer, and later on the case when it is fractional or negative.

I. Positive Integral Exponent.

167. Binomial Theorem for a positive integral exponent.

To prove that

$$(1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + \cdots + {}^nC_{n-1}x^{n-1} + {}^nC_nx^n,$$

where n is any positive integer.

By actual multiplication, we have

$(1+x)^2 = 1+2x+x^2 = 1+{}^2C_1x+{}^2C_2x^2.$

$(1+x)^3 = 1+3x+3x^2+x^3 = 1+{}^3C_1x+{}^3C_2x^2+{}^3C_3x^3.$

$(1+x)^4 = 1+4x+6x^2+4x^3+x^4$

$$= 1+{}^4C_1x+{}^4C_2x^2+{}^4C_3x^3+{}^4C_4x^4.$$

From the results thus obtained, we may infer that

$(1+x)^n = 1+{}^nC_1x+{}^nC_2x^2+\cdots+{}^nC_{n-1}x^{n-1}+{}^nC_nx^n,$

where n is any positive integer. We shall now prove that our inference is true for all positive integral values of n by the method of *Mathematical Induction*. [Art. 142.]

Let us suppose that the theorem is true when $n=m$; then we have

$$(1+x)^m = 1+{}^mC_1x+{}^mC_2x^2+\cdots+{}^mC_{m-1}x^{m-1}+{}^mC_mx^m.$$

Multiplying both sides by $1+x$, we have

$$(1+x)^{m+1}=(1+x)\times(1+{}^mC_1x+{}^mC_2x^2+{}^mC_3x^3 + \ldots + {}^mC_{m-1}x^{m-1}+{}^mC_mx^m)$$

$$=1+({}^mC_1+1)x+({}^mC_2+{}^mC_1)x^2+({}^mC_3+{}^mC_2)x^3 + \ldots + ({}^mC_m+{}^mC_{m-1})x^m+{}^mC_mx^{m+1}.$$

Now, since ${}^mC_r+{}^mC_{r-1}={}^{m+1}C_r$, [Art. 154]
putting $r=1, 2, 3, \ldots\ldots m$, we have

$${}^mC_1+1={}^{m+1}C_1,\ {}^mC_2+{}^mC_1={}^{m+1}C_2,\ {}^mC_3+{}^mC_2={}^{m+1}C_3, \ldots$$
$${}^mC_m+{}^mC_{m-1}={}^{m+1}C_m \text{ and also, } {}^mC_m={}^{m+1}C_{m+1}=1,$$

$$\therefore (1+x)^{m+1}=1+{}^{m+1}C_1x+{}^{m+1}C_2x^2+{}^{m+1}C_3x^3+\ldots + {}^{m+1}C_mx^m+{}^{m+1}C_{m+1}x^{m+1}.$$

Thus, we see that the successive terms of the series for $(1+x)^{m+1}$ are of the same form as was assumed for the terms of $(1+x)^m$, but with $m+1$ written for m; and therefore if the theorem is assumed to be true when n has any positive integral value m, it is also true when $n=m+1$.

But by actual multiplication we have found that the theorem holds in the case of $(1+x)^4$, therefore it holds in the case of $(1+x)^5$; since it holds in the case of $(1+x)^5$, it holds in the case of $(1+x)^6$; and so on. Hence the theorem holds universally; and we have

$$(1+x)^n=1+{}^nC_1x+{}^nC_2x^2+{}^nC_3x^3+ \ldots +{}^nC_{n-1}x^{n-1}+{}^nC_nx^n,$$

for all positive integral values of n.

Since ${}^nC_n=1$, we may as well write

$$(1+x)^n=1+{}^nC_1x+{}^nC_2x^2+{}^nC_3x^3+\ldots+{}^nC_{n-1}x^{n-1}+x^n$$
$$=1+nx+\frac{n(n-1)}{\lfloor 2}x^2+\frac{n(n-1)(n-2)}{\lfloor 3}x^3+\cdots+nx^{n-1}+x^n.$$

writing the values of ${}^nC_1, {}^nC_2, {}^nC_3 \ldots$.

This theorem is known as the *Binomial Theorem for a positive integral exponent.*

Alternative Proof :

Since $(1+x)^n=(1+x)(1+x)(1+x)\cdots\cdots$to n factors, we can easily form the successive terms in the expansion of $(1+x)^n$ from the product of the n factors in the following way :

The term independent of x, *i.e.*, the 1st term is obtained by multiplying together the 1st terms of the n binomial factors, and therefore it is equal to 1.

The term containing the highest power of x, *i. e.*, the last term is obtained by multiplying together all the second terms of the n factors, and therefore it is equal to x^n.

Any term containing x^r is obtained by multiplying together r x's taken out of any r factors and $n-r$ 1's out of the remaining factors; and since r x's can be selected out of the n x's in ${}^nC_r$ ways, there must be ${}^nC_r$ terms each containing x^r, and therefore the co-efficient of x^r in the product of the n factors must be ${}^nC_r$.

Hence, putting $r=1, 2, 3, \cdots$ we have the co-efficient of x $={}^nC_1$, that of $x^2={}^nC_2$, that of $x^3={}^nC_3$, and so on.

Hence $(1+x)^n=1+{}^nC_1x+{}^nC_2x^2+{}^nC_3x^3+\cdots+x^n$, which is the required expansion.

168. In the above expansion for $(1+x)^n$, it is easy to see that

(i) the number of terms in the expansion is $n+1$, *i. e.*, one more than the exponent of $1+x$;

(ii) the first term is always unity and the successive terms contain x in its ascending powers, the last term being x^n;

(iii) the successive co-efficients are ${}^nC_1$, ${}^nC_2$, ${}^nC_3, \cdots {}^nC_n$, and these are called the *binomial co-efficients.*

169. Expansion of $(1-x)^n$, $(a+x)^n$ and $(a-x)^n$.

From the expansion of $(1+x)^n$, may be deduced the expansions of $(1-x)^n$, $(a+x)^n$ and $(a-x)^n$.

Putting $-x$ for x, we have

$$(1-x)^n=1+{}^nC_1.(-x)+{}^nC_2.(-x)^2+{}^nC_3.(-x)^3+\cdots+{}^nC_n(-x)^n$$
$$=1-{}^nC_1x+{}^nC_2x^2-{}^nC_3x^3+\cdots+(-1)^n.x^n$$

i.e., in the expansion of $(1-x)^n$, *the terms are alternately positive and negative, and the last term is* $+x^n$ *or* $-x^n$, *according as* n *is even or odd.*

Again,

$$(a+x)^n=\left\{a\left(1+\frac{x}{a}\right)\right\}^n=a^n\left(1+\frac{x}{a}\right)^n$$
$$=a^n\left\{1+{}^nC_1.\frac{x}{a}+{}^nC_2.\frac{x^2}{a^2}+{}^nC_3.\frac{x^3}{a^3}+\cdots+\frac{x^n}{a^n}\right\}$$
$$=a^n+{}^nC_1a^{n-1}x+{}^nC_2a^{n-2}x^2+{}^nC_3a^{n-3}x^3+\cdots+x^n,$$

Putting $-x$ for x, we have

$$(a-x)^n=a^n-{}^nC_1a^{n-1}x+{}^nC_2a^{n-2}x^2-\cdots+(-1)^nx^n.$$

Example 1. *Expand* $(x+2y)^6$.

$$(x+2y)^6 = x^6 + {}^6C_1x^5(2y) + {}^6C_2x^4(2y)^2 + {}^6C_3x^3(2y)^3 + {}^6C_4x^2(2y)^4 + {}^6C_5x(2y)^5 + {}^6C_6(2y)^6$$
$$= x^6 + 6.\ 2.\ x^5y + + \tfrac{6.5}{1.2}.\ 2^2x^4y^2 + \tfrac{6.5.4}{1.2.3}.\ 2^3x^3y^3 + \tfrac{6.5}{1.2}.\ 2^4x^2y^4 + 6.\ 2^5xy^5 + 2^6y^6$$
$$= x^6 + 12x^5y + 60x^4y^2 + 160x^3y^3 + 240x^2y^4 + 192xy^5 + 64y^6.$$

Example 2. *Expand* $(2x-3y)^7$.

$$(2x-3y)^7 = (2x)^7 - {}^7C_1(2x)^6(3y) + {}^7C_2(2x)^5(3y)^2 - {}^7C_3(2x)^4(3y)^3 + {}^7C_4(2x)^3(3y)^4 - {}^7C_5(2x)^2(3y)^5 + {}^7C_6(2x)(3y)^6 - {}^7C_7(3y)^7$$
$$= 2^7x^7 - 7.2^6.3x^6y + \tfrac{7.6}{1.2}.\ 2^5.3^2x^5y^2 - \tfrac{7.6.5}{1.2.3}.\ 2^4.3^3x^4y^3 + \tfrac{7.6.5}{1.2.3}.\ 2^3.3^4.x^3y^4 - \tfrac{7.6}{1.2}\ 2^2.3^5x^2y^5 + 7.2.3^6xy^6 - 3^7y^7.$$
$$= 128x^7 - 1344x^6y + 6048x^5y^2 - 15120x^4y^3 + 22680x^3y^4 - 20412x^2y^5 + 10206xy^6 - 2187y^7.$$

Example 3. *Find the value of* $(x+\sqrt{x^2-1})^7 + (x-\sqrt{x^2-1})^7$.

In the two expansions in the given expression, the terms are numerically the same; but in the first expansion all the terms are positive, and in the second expansion the second, fourth, sixth and eight terms are negative, and therefore these terms destroy the corresponding terms of the first expansion. Hence, we have the given expression

$$= 2\{x^7 + {}^7C_2x^5(\sqrt{x^2-1})^2 + {}^7C_4x^3(\sqrt{x^2-1})^4 + {}^7C_6x(\sqrt{x^2-1})^6\}$$
$$= 2\{x^7 + 21x^5(x^2-1) + 35x^3(x^2-1)^2 + 7x(x^2-1)^3\}$$
$$= 128x^7 - 224x^5 + 112x^3 - 14x.$$

170. The general or the $(r+1)$th term.

In the expansion of $(1+x)^n$, we have

the 2nd term $= {}^nC_1x$,
,, 3rd term $= {}^nC_2x^2$,
,, 5th term $= {}^nC_4x^4$,
,, 10th term $= {}^nC_9x^9$, and so on.

Hence the $(r+1)$th term $= {}^nC_rx^r$.

Thus the general or the $(r+1)$th term in the expansion of $(1+x)^n = {}^nC_r.x^r. = \dfrac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r}x^r.$

Putting $-x$ for x, the general or the $(r+1)$th term in the expansion of $(1-x)^n$

$$= {}^nC_r(-x)^r = (-1)^r.{}^nC_rx^r$$
$$= (-1)^r\frac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r}x^r.$$

Again in the expansion of $(a+x)^n$, we have

the 2nd term $= {}^nC_1 a^{n-1}x$,

,, 3rd term $= {}^nC_2 a^{n-2}x^2$,

,, 5th term $= {}^nC_4 a^{n-4}x^4$,

,, 10th term $= {}^nC_9 a^{n-9}x^9$, and so on.

Hence the $(r+1)$th term $= {}^nC_r\, a^{n-r}x^r$.

Thus in the expansion of $(a+x)^n$, the general or the $(r+1)$th term $= {}^nC_r a^{n-r}x^r = \dfrac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r}a^{n-r}x^r$.

Putting $-x$ for x, the general or the $(r+1)$th term in the expansion of $(a-x)^n$

$$= {}^nC_r a^{n-r}.(-x)^r = (-1)^r.{}^nC_r a^{n-r}x^r$$
$$= (-1)^r \frac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r}a^{n-r}x^r.$$

Example 1. *Find the 9th term of* $(a-2x)^{15}$.

Here $r+1=9$. $\therefore\ r=8$.

Hence the required term $=(-1)^8\ {}^{15}C_8 a^{15-8}(2x)^8$.

$$= \frac{\lfloor 15}{\lfloor 8 \lfloor 7}\ 2^8 a^7 x^8.$$

Example 2. *Write down the co-efficients of* x^{17} *and* x^{18} *in the expansion of* $(a^4-bx^3)^{10}$. [*Cal. F. A.* 1876].

The $(r+1)$th term $=(-1)^r.{}^{10}C_r(a^4)^{10-r}.(bx^3)^r$

$$=(-1)^r.\ {}^{10}C_r a^{40-4r}b^r x^{3r}.$$

In order that this may contain x^{17}, we must have $3r=17$, *i.e.*, $r=\frac{17}{3}$ which is not an integer. Hence there is no term containing x^{17}, and the co-efficient of x^{17} is 0.

In order that x^{18} may occur in the $(r+1)$th term, we must have $3r=18$, *i.e.*, $r=6$. Hence the 7th term contains x^{18} and the required co-efficient $=(-1)^6.\ {}^{10}C_6 a^{40-24}b^6 = 210a^{16}b^6$.

Example 3. *Find the term independent of x in the expansion of* $\left(x^2+\frac{1}{x^3}\right)^{25}$

Let the $(r+1)$th term be independent of x.

The $(r+1)$th term $= {}^{25}C_r x^{2(25-r)} \cdot \frac{1}{x^{3r}} = {}^{25}C_r . x^{50-5r}$.

In order that it may be independent of x, we must have

$$50-5r=0, \quad \text{or, } r=10.$$

Hence the 11th term is independent of x and it is

$$= {}^{25}C_{10} = \frac{\lfloor 25}{\lfloor 10 \lfloor 15}.$$

Example 4. *If x^p occurs in the expansion of* $\left(x^2+\frac{1}{x}\right)^{2n}$, *prove that its co-efficient is* $\dfrac{\lfloor 2n}{\lfloor \frac{1}{3}(4n-p) \lfloor \frac{1}{3}(2n+p)}$

Let x^p occur in the $(r+1)$th term.

The $(r+1)$th term $= {}^{2n}C_r (x^2)^{2n-r}\left(\frac{1}{x}\right)^r = {}^{2n}C_r x^{4n-3r}$.

Hence, in order that x^p may occur in the $(r+1)$th term, we must have $4n-3r=p$. $\therefore$ $r=\frac{1}{3}(4n-p)$.

Therefore in the given expansion, there will be a term containing x^p if $\frac{1}{3}(4n-p)$ be an integer, *i.e.*, if $4n-p$ be a multiple of 3, and the required co-efficient will be

$$= {}^{2n}C_r = {}^{2n}C_{\frac{1}{3}(4n-p)} = \frac{\lfloor 2n}{\lfloor \frac{1}{3}(4n-p) \lfloor \frac{1}{3}(2n+p)}.$$

171. Equidistant Terms.

To prove that in the expansion of $(1+x)^n$ the co-efficients of terms equidistant from the beginning and the end are equal.

We know that the co-efficient of the $(r+1)$th term from the beginning $= {}^nC_r$. [Art. 170]

Since the number of terms in the expansion of $(1+x)^n$ is $n+1$, it is evident that the $(r+1)$th term from the end is the same as the $(n-r+1)$th term from the beginning and the co-efficient of the $(n-r+1)$th term $= {}^nC_{n-r}$.

But $\quad {}^nC_r = {}^nC_{n-r}$. [Art. 153]

$\therefore$ the co-efficient of the $(r+1)$th term from the beginning = the co-efficient of the $(r+1)$th term from the end.

172. Middle Term.

To find the middle term in the expansion of $(1+x)^n$.

(i) Let n be *even* and equal to $2m$, so that $m=\frac{1}{2}n$.

Since the total number of terms is $2m+1$ (an odd number), there will be only *one middle term*, and the $(m+1)$th term, *i.e.*, the $(\frac{1}{2}n+1)$th term is the middle term.

Hence the middle term $= {}^nC_{\frac{n}{2}}x^{\frac{n}{2}} = \dfrac{\underline{|n}}{\underline{|\frac{1}{2}n}\;\underline{|\frac{1}{2}n}}\,x^{\frac{n}{2}}$.

(ii) Let n be *odd* and equal to $2m+1$, so that $m=\frac{1}{2}(n-1)$.

Since the total number of terms is $2m+2$ (an even number), there will be *two middle terms*, and the $(m+1)$th and $(m+2)$th terms *i.e.*, $(\frac{n-1}{2}+1)$th and $(\frac{n+1}{2}+1)$th terms are the two middle terms. Hence the two middle terms are

$$ {}^nC_{\frac{n-1}{2}}x^{\frac{n-1}{2}} \text{ and } {}^nC_{\frac{n+1}{2}}x^{\frac{n+1}{2}} $$

i.e., $$\frac{\underline{|n}}{\underline{|\frac{1}{2}(n-1)}\;\underline{|\frac{1}{2}(n+1)}}\,x^{\frac{n-1}{2}} \text{ and } \frac{\underline{|n}}{\underline{|\frac{1}{2}(n+1)}\;\underline{|\frac{1}{2}(n-1)}}\,x^{\frac{n+1}{2}}.$$

Example 1. *Obtain the middle term of* $\left(x-\dfrac{1}{x}\right)^{10}$

[*Cal. Int.* 1910.]

Since there are 11 terms in the expansion, the 6th term is the middle term, which

$$=(-1)^5.{}^{10}C_5x^{10-5}\cdot\frac{1}{x^5}=(-1)^5\,\frac{\underline{|10}}{\underline{|5}\,\underline{|5}}.\,x^5.\,\frac{1}{x^5}=-252.$$

Example 2. *Show that the middle term in the expansion of* $(1+x)^{2n}$ *is* $\dfrac{1.3.5\ldots\ldots(2n-1)}{\underline{|n}}\,2^nx^n$.

Since there are $(2n+1)$ terms in the expansion, the $(n+1)$th term is the middle term, which

$$= {}^{2n}C_nx^n=\frac{\underline{|2n}}{\underline{|n}\,\underline{|n}}x^n=\frac{1.2.3.4.5\ldots\ldots(2n-1).2n}{\underline{|n}\,\underline{|n}}x^n$$

$$=\frac{1.3.5\ldots\ldots(2n-1)2.4.6\ldots2n}{\underline{|n}\,\underline{|n}}x^n$$

$$=\frac{1.3.5\ldots\ldots(2n-1)2^n.\underline{|n}}{\underline{|n}\,\underline{|n}}x^n$$

$$=\frac{1.3.5\ldots\ldots(2n-1)}{\underline{|n}}2^nx^n.$$

EXERCISE 57.

Expand :

1. $(1+x)^6$.	2. $(a+b)^7$.	3. $(x+2)^5$.
4. $(2x-1)^5$.	5. $(2x+3)^4$.	6. $(a^2-bc)^6$.
7. $(x-2y)^8$.	8. $(1+\frac{1}{2}x)^7$.	9. $(\frac{3}{2}x-\frac{2}{3}y)^5$.

10. Find the 5th term of $(1+2x)^{10}$.

11. Find the 9th term in the expansion of $(\frac{1}{4}a-\frac{1}{2}b)^{12}$. [Cal. F. A. 1868]

12. Write down the 4th term of $\left(\frac{x}{a}-\frac{a}{x}\right)^{10}$. [Cal. F. A. 1888]

13. Write down the 6th term of $(3x+\frac{1}{2}y)^9$.

14. Write down the 9th term in the expansion of $(2x^{\frac{1}{2}}-y^{\frac{1}{3}})^{20}$. [Cal. F. A. 1870]

15. Write down the general term in $\left(x+\frac{1}{x}\right)^n$.

Expand and simplify :

16. $(1+\sqrt{5})^5+(1-\sqrt{5})^5$. 17. $(\sqrt{2}+1)^6-(\sqrt{2}-1)^6$.

18. $(x-\sqrt{1-x^2})^4+(x+\sqrt{1-x^2})^4$.

19. $(3+2\sqrt{x})^6+(3-2\sqrt{x})^6$.

20. Find the middle term of $(x-2y)^8$.

21. Find the middle term of $\left(x^2 \quad 1\right)^6$

22. Find the two middle terms of $\left(\frac{x}{y}-\frac{y}{x}\right)^7$.

23. Find the two middle terms of $(a+x)^{2n+1}$.

24. Expand $\left(\frac{a}{b}+\frac{b}{a}\right)^{2n+1}$, giving in particular the general term and the two middle terms. [Cal. Int. 1932]

25. Find the co-efficient of x^{11} in $(x-2y)^{13}$. [Cal. F. A. 1883]

26. Find the co-efficient of x^{15} in the expansion of $(x-x^2)^{10}$. [Cal. Int. 1926]

27. Find the co-efficient of $\frac{1}{y^2}$ in $\left(y+\frac{c^3}{y^2}\right)^{10}$. [Pun. I. E. 1887]

28. What is the co-efficient of x^m in the expansion of $(x+y)^n$? [Cal. F. A. 1876]

29. Find the co-efficient of x^{2r+1} in the expansion of $\left(x-\dfrac{1}{x}\right)^{2n+1}$

30. Find the term independent of x in $\left(ax^3-\dfrac{b}{x^4}\right)^{35}$

31. Find the term independent of x in the expansion of $\left(2x+\dfrac{1}{3x^2}\right)^9$. [Cal. Int. 1936]

32. Find the term independent of x in the expansion of $\left(x^2+\dfrac{1}{x}\right)^{12}$. [Cal. Int. 1934]

33. Obtain the term free from x in the expansion of $\left(x+\dfrac{1}{x}\right)^{2n}$. [Cal. Int. 1931]

34. If there is a term independent of x in $\left(x+\dfrac{1}{x}\right)$, show that it is equal to $\underline{|n}$.

35. If x^r occurs in the expansion of $\left(x+\dfrac{1}{x}\right)^n$, show that its co-efficient is $\dfrac{\underline{|n}}{\underline{|\tfrac{1}{2}(n-r)}\,\underline{|\tfrac{1}{2}(n+r)}}$.

36. Show that the co-efficient of the middle term of $(1+x)^{2n}$ is equal to the sum of the co-efficients of the two middle terms of $(1+x)^{2n-1}$.

37. Show that the co-efficient of the $(r+1)$th term in the expansion of $(1+x)^{n+1}$ is equal to the sum of the co-efficients of the rth and $(r+1)$th terms in that of $(1+x)^n$. [Cal. F. A. 1892]

38. If in the expansion of $(1+x)^{2n+1}$ the co-efficients of x^r and x^{r+1} be equal, find r, [Cal. Int. 1930]

39. In the expansion of $(1+x)^{m+n}$, where m and n are positive integers, prove that the co-efficients of x^m and x^n are equal. [Cal. Int. 1935]

40. In the expansion of $(x^a+x^{-b})^n$, show that there is no term independent of x unless an is an integral multiple of $a+b$.

173. Greatest Co-efficient.

To find the greatest co-efficient in the expansion of $(1+x)^n$.

Since $(1+x)^n = 1 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + \cdots\cdots + {}^nC_nx^n$, the co-efficients are ${}^nC_1, {}^nC_2, {}^nC_3 \ldots {}^nC_n$, and we require to find which of these is the greatest. The co-efficient of the $(r+1)$th term is ${}^nC_r$.

We know from Art. 156, that if n be even, ${}^nC_r$ is greatest when $r = \frac{n}{2}$; and if n be odd, ${}^nC_r$ is greatest when $r = \frac{n-1}{2}$ or $\frac{n+1}{2}$.

Hence, if n be even, the greatest co-efficient is that of the $\left(\frac{n}{2}+1\right)$th term, namely, ${}^nC_{\frac{n}{2}}$, and if n be odd, the greatest co-efficient is that of the $\left(\frac{n-1}{2}+1\right)$th or $\left(\frac{n+1}{2}+1\right)$th term, namely, ${}^nC_{\frac{n-1}{2}}$ or ${}^nC_{\frac{n+1}{2}}$, which are equal.

174. Greatest Term.

To find the greatest term in the expansion of $(a+x)^n$.

The rth term $= \frac{n(n-1)(n-2)\cdots\cdots(n-r+2)}{1.2.3.\cdots(r-1)} a^{n-r+1} x^{r-1}$, and

the $(r+1)$th term $= \frac{n(n-1)(n-2)\ldots(n-r+1)}{1.2.3.\cdots(r-1)r} a^{n-r} x^r$.

Hence, the $(r+1)$th term is obtained from the rth term by multiplying it by $\frac{n-r+1}{r}.\frac{x}{a}$.

Therefore if we denote the rth term and the $(r+1)$th term by T_r and T_{r+1} respectively, we have

$$T_{r+1} = \frac{n-r+1}{r} \cdot \frac{x}{a} \times T_r = \left(\frac{n+1}{r} - 1\right)\frac{x}{a} \times T_r.$$

Thus, $T_{r+1} >, =$, or $< T_r$, according as the multiplying factor $\left(\frac{n+1}{r}-1\right)\frac{x}{a} >, =$, or <1. Now as r increases, the multiplying factor $\left(\frac{n+1}{r}-1\right)\frac{x}{a}$ diminishes, and as r

receives the integral values from 1 to n, the multiplying factor $\left(\frac{n+1}{r}-1\right)\frac{x}{a}$ gradually diminishes from $\frac{nx}{a}$ to $\frac{x}{na}$. Hence, the terms of the series $T_1, T_2, T_3, \cdots\cdots T_{r+1}$ continue to increase from the beginning so long as the value of r is such that $\left(\frac{n+1}{r}-1\right)\frac{x}{a}$ is greater than 1, and after this as soon as it becomes less than 1 the terms of the series continue to diminish as r still increases. Hence T_{r+1} ceases to increase and then attains its greatest value when the multiplying factor $\left(\frac{n+1}{r}-1\right)\frac{x}{a}$ is either just $>$ or $=1$, that is to say, T_{r+1} is greatest

when $\frac{n+1}{r}-1$ is either just $>$ or $=\frac{a}{x}$

i.e., " $\frac{n+1}{r}$ " " $>$ or $=\frac{a}{x}+1$

i.e., " $\frac{n+1}{\frac{a}{x}+1}$ " " $>$ or $=r$

i.e., " r " " $<$ or $=\frac{n+1}{\frac{a}{x}+1}$

(i) If $\frac{n+1}{\frac{a}{x}+1}$ be an integer, let us denote it by p.

Then, since r must be an integer, in order that T_{r+1} may be greatest we must have $r=p$, and in this case the multiplying factor is equal to 1. Hence the pth and the $(p+1)$th terms are equal and both of them are the greatest terms.

(ii) If $\frac{n+1}{\frac{a}{x}+1}$ be not an integer, let us denote its integral part by q. Then since q is the integer just less than $\frac{n+1}{\frac{a}{x}+1}$ and r must be an integer, in order that T_{r+1} is greatest, we must have $r=q$, and therefore the $(q+1)$th term is the greatest.

Since the terms in the expansion of $(a-x)^n$ are numerically the same as the corresponding terms in the expansion of $(a+x)^n$, in finding the numerically greatest term in the expansion of $(a-x)^n$ we ignore the negative sign and proceed exactly in the same way as in the case of $(a+x)^n$.

Example 1. *Find the greatest term in the expansion of* $(2+3x)^{13}$, *when* $x=\frac{1}{2}$.

Let the $(r+1)$th term be the greatest.

Now $T_{r+1}=\frac{13-r+1}{r}\cdot\frac{3x}{2}\times T_r=\left(\frac{14}{r}-1\right)\frac{3}{4}\times T_r$.

$\therefore$ T_{r+1} is greatest when the multiplying factor

$$\left(\frac{14}{r}-1\right)\frac{3}{4} \text{ is either just} > \text{or} = 1$$

i.e., when $\frac{14}{r}$ „ $>$ or $=\frac{4}{3}+1$

i.e., „ 14 „ $>$ or $=\frac{7}{3}r$

i.e., „ r „ $<$ or $=\frac{14\times 3}{7}$, *i.e.*, 6.

Since r must be an integer, we have in this case $r=6$, and therefore the 6th and 7th terms are numerically equal to each other and both of them are the greatest terms.

And the numerical value of the greatest term

$$={}^{13}C_5 . 2^8 . (3x)^5=\frac{13\cdot 12\cdot 11\cdot 10\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 5}.2^8.\left(\tfrac{3}{2}\right)^5$$
$$=13\times 11\times 9\times 8\times 243=2501928.$$

Example 2. *Find the greatest numerical co-efficient in the expansion of* $(3x+5y)^{10}$.

It is obvious that the greatest numerical co-efficient in the expansion of $(3x+5y)^{10}$ is the same as the greatest term in the expansion of $(3+5)^{10}$.

Let the $(r+1)$th term in $(3+5)^{10}$ be the greatest.

Now $T_{r+1}=\frac{10-r+1}{r}\cdot\frac{5}{3}\times T_r=\left(\frac{11}{r}-1\right)\frac{5}{3}\times T_r$

T_{r+1} is greatest when the multiplying factor

$$\left(\frac{11}{r}-1\right)\frac{5}{3} \text{ is either just} > \text{or} = 1$$

i.e., when $\frac{11}{r}$ „ $>$ or $=\frac{3}{5}+1$, *i.e.*, $\frac{8}{5}$

i.e., „ 11 „ $>$ or $=\frac{8}{5}r$

i.e., „ r „ $<$ or $=\frac{55}{8}$ *i.e.*, $6\frac{7}{8}$.

Now since r must be an integer just less than $6\frac{7}{8}$, we have $r=6$, and therefore the 7th term is the greatest; and its numerical value $= {}^{10}C_6 . 3^4 . 5^6 = \frac{10.9.8.7}{1.2.3.4} . 81 \ . 15625 = 267581250$.

EXERCISE 58.

Find the greatest terms in the expansion of :

1. $(2+3)^{14}$. 2. $(1+\frac{5}{8})^{12}$.

3. $(1+x)^6$, when $x=\frac{1}{2}$. 4. $(x+y)^{13}$, when $x=5$, $y=2$. [Bom. P. E. 1885].

5. $(1+2x)^9$, when $x=\frac{1}{3}$. 6. $(5a-3x)^9$, when $x=\frac{3}{2}$ and $a=\frac{2}{3}$.

7. Find the numerically greatest co-efficient in the expansion of $(1+\frac{2}{3}x)^{12}$.

8. Find the greatest numerical co-efficient in the expansion of $\left(3ax-\frac{2b^3}{x}\right)^{14}$.

9. Find the greatest term in the expansion of $\left(n-\frac{1}{n}\right)^{2n+1}$, when n is a positive integer.

10. Find the limits between which x must lie in order that the greatest term in the expansion of $(1+x)^{30}$ may have the greatest co-efficient.

11. Prove that if x lies between $\frac{n}{n+2}$ and $\frac{n+2}{n}$, the greatest term in the expansion of $(1+x)^{2n+1}$ has the greatest co-efficient. [Mad. F. A. 1903]

12. Show that if x lies between $\frac{n}{n+1}$ and $\frac{n+1}{n}$, the greatest term in the expansion of $(1+x)^{2n}$ has also the greatest co-efficient. [Mad. F. A. 1898]

175. Binomial Co-efficients and their properties.

For the sake of brevity, the binomial co-efficients, ${}^nC_0$, ${}^nC_1$, ${}^nC_2, \cdots {}^nC_n$ are sometimes denoted by the symbols $C_0, C_1, C_2, \cdots C_n$, so that we may write

$$(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n.$$

Also since ${}^nC_r = {}^nC_{n-r}$, we have

$C_0 = C_n = 1$, $C_1 = C_{n-1} = n$, $C_2 = C_{n-2} = \frac{n(n-1)}{1.2}$, and so on.

We prove below some useful properties of the binomial co-efficients.

(i) *The sum of the binomial co-efficients* $C_0, C_1, C_2, \cdots C_n$ *is equal to* 2^n.

Since $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$,
we have putting $x=1$, $2^n = C_0 + C_1 + C_2 + \cdots + C_n$.

i.e., the sum of the binomial co-efficients $=2^n$.

Since $C_0 = 1$, we have

$$C_1 + C_2 + C_3 + \cdots + C_n = 2^n - 1,$$

already ~~proved in the previous chapter [Art. 160]~~.

(ii) *The sum of the even binomial co-efficients is equal to the sum of the odd ones ; i.e.*,

$$C_0 + C_2 + C_4 + \cdots = C_1 + C_3 + C_5 + \cdots\cdots$$

In the expansion $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$, putting $x = -1$, we have

$$0 = C_0 - C_1 + C_2 - C_3 + \cdots + (-1)^n C_n$$
$$= (C_0 + C_2 + C_4 + \cdots) - (C_1 + C_3 + C_5 + \cdots)$$
$$\therefore\ C_0 + C_2 + C_4 + \cdots = C_1 + C_3 + C_5 + \cdots$$

Since the sum of all the co-efficients $=2^n$, each of these equal expressions $= \frac{1}{2}.\, 2^n = 2^{n-1}$.

[Exam]ple 1. *If* $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots\cdots\cdots + C_nx^n$,
prove that $C_0^2 + C_1^2 + C_2^2 + \cdots + C_n^2 = \dfrac{\lfloor 2n}{\lfloor n \lfloor n}$. [*Cal. Int.* 1933]

We have $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots\cdots + C_nx^n$,

[a]nd $(x+1)^n = C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots\cdots + C_n$.

Hence, on multiplication,

$$(1+x)^{2n} = (C_0 + C_1x + C_2x^2 + \cdots\cdots + C_nx^n)$$
$$\times (C_0x^n + C_1x^{n-1} + C_2x^{n-2} + \cdots\cdots + C_n),$$

This being an identity, the co-efficient of x^n on the left-hand side must be equal to the co-efficient of x^n on the right-hand side.

The coefficient of x^n in the product of the two expressions on the right-hand side is evidently

$$= C_0^2 + C_1^2 + C_2^2 + \cdots\cdots + C_n^2.$$

And the co-eff. of x^n in the expansion of $(1+x)^{2n} = {}^{2n}C_n = \frac{\lfloor 2n}{\lfloor n \lfloor n}$.

$$\therefore \qquad C_0{}^2 + C_1{}^2 + C_2{}^2 + \cdots + C_n{}^2 = \frac{\lfloor 2n}{\lfloor n \lfloor n}.$$

Example 2. *Prove that*

$C_0 + 2C_1 + 3C_2 + \cdots\cdots + (n+1)C_n = (n+2)2^{n-1}$. [*Cal. Int.* 1929.]

$$\begin{aligned} C_0 + 2C_1 + 3C_2 + \cdots\cdots + (n+1)C_n & \\ &= (C_0 + C_1 + C_2 + \cdots\cdots + C_n) + (C_1 + 2C_2 + \cdots\cdots + nC_n) \\ &= 2^n + \left\{ n + 2.\frac{n(n-1)}{\lfloor 2} + 3.\frac{n(n-1)(n-2)}{\lfloor 3} + \cdots\cdots + n \right\} \\ &= 2^n + n\left\{1 + (n-1) + \frac{(n-1)(n-2)}{\lfloor 2} + \cdots\cdots + 1\right\} \\ &= 2^n + n\{1+1\}^{n-1} \\ &= 2^n + n.2^{n-1} \\ &= (n+2).2^{n-1}. \end{aligned}$$

Example 3. *If* $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots\cdots\cdots\cdots\cdots + C_nx^n$, *find the value of* $\quad \frac{C_0}{1} + \frac{C_2}{3} + \frac{C_4}{5} + \frac{C_6}{7} + \cdots\cdots$

Since $(1+x)^{n+1} = 1 + {}^{n+1}C_1x + {}^{n+1}C_2x^2 + \cdots\cdots + {}^{n+1}C_{n+1}x^{n+1}$ we have $1 + {}^{n+1}C_2 + {}^{n+1}C_4 + \cdots = {}^{n+1}C_1 + {}^{n+1}C_3 + {}^{n+1}C_5 + \cdots = 2^n$.

Now ${}^{n+1}C_1 = n+1 = (n+1)C_0$,

$${}^{n+1}C_3 = \frac{(n+1)n(n-1)}{1.2.3} = (n+1)\frac{C_2}{3},$$

$${}^{n+1}C_5 = \frac{(n+1)n(n-1)(n-2)(n-3)}{1.2.3.4.5} = (n+1)\frac{C_4}{5}\text{ ; etc.}$$

$$\therefore \qquad (n+1)C_0 + (n+1)\frac{C_2}{3} + (n+1)\frac{C_4}{5} + \cdots\cdots = 2^n.$$

Hence dividing by $(n+1)$, we have

$$\therefore\ C_0 + \frac{C_2}{3} + \frac{C_4}{5} + \frac{C_6}{7} + \cdots\cdots = \frac{2^n}{n+1}.$$

EXERCISE 59.

If $(1+x)^n = C_0 + C_1x + C_2x^2 + \cdots + C_nx^n$, prove that

1. $C_1 + 2C_2 + 3C_3 + \cdots\cdots + nC_n = n.2^{n-1}$. [Cal. Int. 1938]

2. $C_0 + 3C_1 + 5C_2 + \cdots\cdots + (2n+1)C_n = (n+1)2^n$.

3. $C_1 - 2C_2 + 3C_3 - \cdots\cdots + (-1)^{n-1}.nC_n = 0$. [Cal. Int. 1942]

4. $C_0 + \frac{1}{2}C_1 + \frac{1}{3}C_2 + \cdots\cdots + \frac{1}{n+1}C_n = \frac{2^{n+1}-1}{n+1}$.

[Cal. Int. 1945, Mad. F. A. 1881, and All. I. E. 1927]

5. $C_0 - \frac{1}{2}C_1 + \frac{1}{3}C_2 - \ldots\ldots + (-1)^n \frac{1}{n+1}C_n = \frac{1}{n+1}$.

6. $C_0 + C_1x + 2C_2x^2 + \ldots\ldots + nC_nx^n = 1 + nx(1+x)^{n-1}$.

7. $C_0C_1 + C_1C_2 + C_2C_3 + \ldots\ldots + C_{n-1}C_n = \dfrac{\underline{|2n}}{\underline{|n+1}\;\underline{|n-1}}$

8. $C_0C_2 + C_1C_3 + C_2C_4 + \cdots\cdots + C_{n-2}C_n = \dfrac{\underline{|2n}}{\underline{|n+2}\;\underline{|n-2}}$

9. $C_0^2 + 2C_1^2 + 3C_2^2 + \cdots\cdots + (n+1)C_n^2 = \dfrac{(n+2)\underline{|2n-1}}{\underline{|n-1}\;\underline{|n}}$

10. $C_1^2 + 2C_2^2 + 3C_3^2 + \cdots\cdots + nC_n^2 = \dfrac{\underline{|2n-1}}{(\underline{|n-1})^2}$.

11. $\dfrac{C_1}{C_0} + \dfrac{2C_2}{C_1} + \dfrac{3C_3}{C_2} + \cdots\cdots + \dfrac{nC_n}{C_{n-1}} = \frac{1}{2}n(n+1)$.

12. $C_0^2 - C_1^2 + C_2^2 - C_3^2 + \ldots + (-1)^nC_n^2 = 0$ or $(-1)^{\frac{n}{2}}\dfrac{\underline{|n}}{\{\underline{|\tfrac{1}{2}n}\}^2}$

according as n is odd or even.

13. $(C_0 + C_1)(C_1 + C_2)(C_2 + C_3)\cdots\cdots(C_{n-1} + C_n)$

$$= \frac{C_1C_2C_3\cdots\cdots C_n(n+1)^n}{\underline{|n}}.$$

14. $2C_0 + 2^2.\dfrac{C_1}{2} + 2^3.\dfrac{C_2}{3} + \ldots\ldots + 2^{n+1}.\dfrac{C_n}{n+1} = \dfrac{3^{n+1}-1}{n+1}$.

15. $C_0^2 + \frac{1}{2}C_1^2 + \frac{1}{3}C_2^2 + \cdots\cdots + \dfrac{1}{n+1}C_n^2 = \dfrac{\underline{|2n+1}}{\{\underline{|n+1}\}^2}$.

176. We work out below a few miscellaneous examples illustrating the applications of the Binomial Theorem.

Example 1. *If n be a positive integer, prove that the integral part of $(3+\sqrt{5})^n$ is an odd number.*

Let I be the greatest integer in $(3+\sqrt{5})^n$, and F the remainder which is evidently a fraction less than 1.

Then $I+F=(3+\sqrt{5})^n$

$$=3^n+n.3^{n-1}.5^{\frac{1}{2}}+\frac{n(n-1)}{\lfloor 2}.3^{n-2}.5+\cdots\quad\ldots\quad(1)$$

Since $\sqrt{5}$ lies between 2 and 3, $3-\sqrt{5}$ is a fraction less than 1, and therefore $(3-\sqrt{5})^n$ is also a fraction less than 1. Denoting this by F', we have

$$F'=(3-\sqrt{5})^n=3^n-n\,3^{n-1}.5^{\frac{1}{2}}+\frac{n(n-1)}{\lfloor 2}.3^{n-1}.5-\quad\ldots\quad(2)$$

Adding (1) and (2),

$$I+F+F'=2\left\{3^n+\frac{n(n-1)}{\lfloor 2}.3^{n-2}.5+\quad\cdots\cdots\right\}$$

But each term of the expression within the brackets on the right-hand side is a positive integer.

$\therefore\quad I+F+F'$ is an even number.

Since F and F' are each of them <1, and $F+F'$ is an integer, it follows that $F+F'=1$.

Hence, $I+F+F'$ being an even number, I must be an odd number.

Example 2. *If in the expansion of $(a+x)^n$, A be the sum of the odd terms and B the sum of the even terms, prove that*

$$A^2-B^2=(a^2-x^2)^n.$$

Denoting the terms in the expansion of $(a+x)^n$ by t_0, t_1, $t_2, \cdots t_n$, we have

$$(a+x)^n=t_0+t_1+t_2+\cdots\cdots+t_n=A+B,$$

and $$(a-x)^n=t_0-t_1+t_2-t_3+\cdots\cdots+(-1)^n t_n=A-B.$$

Hence, on multiplication, we have

$$(A+B)(A-B)=(a+x)^n\times(a-x)^n$$

i.e., $$A^2-B^2=(a^2-x^2)^n.$$

Example 3. *Show that when n is any positive integer other than unity, $6^{2n}-35n-1$ is divisible by 1225.*

Since $6^{2n}=36^n=(1+35)^n$, we have

$$6^{2n}-35n-1=(1+35)^n-35n-1$$
$$=1+n.35+{}^nC_2.35^2+{}^nC_3.35^3+\cdots-35n-1$$
$$={}^nC_2.35^2+{}^nC_3.35^3+\ldots\ldots$$

which is evidently divisible by 35^2, *i. e.*, by 1225.

Example 4. *Find the approximate value of* $(0\cdot999)^7$ *correct to 7 places of decimals.*

$$(0\cdot999)^7=\left(\frac{999}{1000}\right)^7=\left(1-\frac{1}{1000}\right)^7=\left(1-\frac{1}{10^3}\right)^7$$

$$=1-\frac{7}{10^3}+\frac{7.6}{1.2}\cdot\frac{1}{10^6}-\frac{7\cdot6\cdot5}{1.2.3}\cdot\frac{1}{10^9}+\cdots$$

$$=1-\cdot007+\cdot000021-\cdot000000035+\cdots$$

$=\cdot9930210$, correct to 7 decimal places.

Example 5. *If* n_r *represent the co-efficient of the* $(r+1)$*th term in the expansion of* $(1+x)^n$, *prove that*

$$(m+n)_r=m_r+m_{r-1}n_1+m_{r-2}n_2+m_{r-3}n_3+\cdots+m_1n_{r-1}+n_r.$$

Since n_r is the co-efficient of $(r+1)$th term in the expansion $(1+x)^n$, we have

$$(1+x)^n=1+n_1x+n_2x^2+\cdots\cdots+n_{r-1}x^{r-1}+n_rx^r+\cdots$$

and $(1+x)^m=1+m_1x+m_2x^2+\cdots+m_{r-1}x^{r-1}+m_rx^r+\cdots$

and $(1+x)^{m+n}=1+(m+n)_1x+(m+n)_2x^2+\cdots+(m+n)_rx^r+\cdots$

But $\qquad (1+x)^m\times(1+x)^n=(1+x)^{m+n}.$

$$\therefore\quad (1+m_1x+m_2x^2+\cdots+m_{r-1}x^{r-1}+m_rx^r+\cdots)$$
$$\times(1+n_1x+n_2x^2+\cdots+n_{r-1}x^{r-1}+n_rx^r+\cdots)$$
$$=1+(m+n)_1x+(m+n)_2x^2+\cdots+(m+n)_rx^r+\cdots$$

This being an identity, equating the co-efficient of x^r on both sides, we have

$$(m+n)_r=m_r+m_{r-1}n_1+m_{r-2}n_2+m_{r-3}n_3+\cdots+m_1n_{r-1}+n_r.$$

EXERCISE 60.

1. Find the algebraic sum of the numerical co-efficients of the several powers of x in the expansion of $(3-2x)^n$.

2. If a_r denote the co-efficient of x^r in the expansion of $(1-x)^{2n-1}$, prove that $a_{r-1}+a_{2n-r}=0$.

3. If n be any positive integer, prove that the integral part of $(4+\sqrt{11})^n$ is an odd number.

4. If a and n be any positive integers, show that the integral part of $(a+\sqrt{a^2-1})^n$ is an odd number.

5. If three successive co-efficients in the expansion of $(1+x)^n$ be 220, 495 and 792, find n.

6. Show that when n is any positive integer except unity $2^{5n}-31n-1$ is divisible by 961.

7. If $t_0, t_1, t_2, \cdots t_n$ be the terms in the expansion of $(a+x)^n$, prove that $(t_0 - t_2 + t_4 - \cdots)^2 + (t_1 - t_3 + t_5 - \cdots)^2 = (a^2+x^2)^n$.

8. Show that when n is any positive integer, $5^{2n+2} - 24n - 25$ is divisible by 576.

9. Employ the Binomial Theorem to find the approximate values of the following, correct to 5 places of decimals :

(i) $(1{\cdot}002)^5$. (ii) $({\cdot}999)^3$. (iii) $(1{\cdot}0005)^7$.

10. If a, b, c be three consecutive co-efficients in the expansion of a power of $1+x$, prove that the index of the power is $\dfrac{2ac+b(a+c)}{b^2-ac}$, and that the number of the term of which a is the co-efficient is $\dfrac{a(b+c)}{b^2-ac}$. [Bom. P. E. 1890]

11. Show that

$$\left(\frac{1+2x}{1+x}\right)^n = 1 + {}^nC_1 \cdot \frac{x}{1+x} + {}^nC_2 \cdot \left(\frac{x}{1+x}\right)^2 + \cdots + \left(\frac{x}{1+x}\right)^n.$$

12. Show that

$$(1-x^2)^n = (1+x)^{2n} - 2nx(1+x)^{2n-1} + \frac{2n(2n-2)}{\underline{|2}}x^2(1+x)^{2n-2} + \cdots$$

13. Find the value of

$$a - n(a-b) + \frac{n(n-1)}{1.2}(a-2b) - \frac{n(n-1)(n-2)}{1.2.3}(a-3b) + \cdots + (-1)^n(a-nb).$$

14. Show that

$$\left(x+\frac{1}{x}\right)^n = \left(x^n + \frac{1}{x^n}\right) + {}^nC_1\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + {}^nC_2\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \cdots$$

and give the last term.

15. If $(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + \cdots + a_{2n}x^{2n}$, show that

(i) $a_0 + a_1 + a_2 + \cdots + a_{2n} = 3^n$;

(ii) $a_0 - a_1 + a_2 - \cdots + a_{2n} = 1$;

(iii) $a_0 + a_3 + a_6 + \cdots = a_1 + a_4 + a_7 + \cdots = a_2 + a_5 + a_8 + \cdots = 3^{n-1}$.

16. If $a_0, a_1, a_2, \cdots a_n$ be the successive co-efficients in the expansion of $(1+x)^n$, show that

$$(a_0 - a_2 + a_4 - \cdots)^2 + (a_1 - a_3 + a_5 \cdots)^2 = a_0 + a_1 + a_2 + \cdots + a_n.$$

II. Any Exponent

177. Infinite Binominal Series.

In the Binomial expansion

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2}x^2 + \frac{n(n-1)(n-2)}{\lfloor 3}x^3 + \cdots\cdots$$

when n is a positive integer, the number of terms finite and equal to $n+1$ [Art. 168].

But if n is not a positive integer, the co-efficient of the general or the $(r+1)$th term

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r}$$

cannot vanish for any value of r (which must necessarily be a positive integer) Hence, the number of terms in the Binomial Series becomes in this case infinite. We have accordingly to investigate for what values of x, this infinite series is convergent; for, as pointed out in Art. 145, divergent infinite series are useless and even misleading.

178. Convergency of Binomial Series when $x<1$.

In the infinite Binomial Series

$$1 + nx + \frac{n(n-1)}{\lfloor 2}x^2 + \frac{n(n-1)(n-2)}{\lfloor 3}x^3 + \cdots\cdots,$$

let T_{r+1} and T_r denote respectively the $(r+1)$th and rth terms.

Then $$T_{r+1} = \frac{n(n-1)(n-2)\cdots\cdots(n-r+1)}{\lfloor r}x^r$$

and $$T_r = \frac{n(n-1)(n-2)\cdots\cdots(n-r+2)}{\lfloor r-1}x^{r-1}$$

$$\therefore \quad \frac{T_{r+1}}{T_r} = \frac{n-r+1}{r}\cdot x = \left(\frac{n+1}{r} - 1\right)x.$$

As r increases, $\frac{n+1}{r}$ continually decreases numerically whatever value n may have, and therefore $\frac{n+1}{r} - 1$ continually approaches the value -1.

Hence, as r increases indefinitely, the ratio $\left(\frac{n+1}{r} - 1\right)x$ continually approaches x numerically. Hence, if $x<1$ numerically, the series is convergent. [Art. 146]

179. Binomial Theorem for any exponent.

Let the functional symbol $f(m)$ denote the series

$$1+mx+\frac{m(m-1)}{\lfloor 2}x^2+\frac{m(m-1)(m-2)}{\lfloor 3}x^3+\cdots\cdots$$

for all values of m whether positive or negative, integral or fractional; then we have

$$f(m)=1+mx+\frac{m(m-1)}{\lfloor 2}x^2+\quad\cdots\quad\cdots\quad(1)$$

$$f(n)=1+nx+\frac{n(n-1)}{\lfloor 2}x^2+\quad\cdots\quad\cdots\quad(2)$$

and $$f(m+n)=1+(m+n)x+\frac{(m+n)(m+n-1)}{\lfloor 2}x^2+\quad\cdots\quad(3)$$

for all values of m and n. Thus, putting $m=0, 1, 2,\cdots$ we have $f(0)=1$, $f(1)=1+x$, $f(2)=1+2x+x^2=(1+x)^2$, and so on.

It is also obvious that, when m and n are positive integers, $f(m)=(1+x)^m$, $f(n)=(1+x)^n$, and $f(m+n)=(1+x)^{m+n}$.

Now, $$(1+x)^m\times(1+x)^n=(1+x)^{m+n}$$

$$\therefore\quad\left\{1+mx+\frac{m(m-1)}{\lfloor 2}x^2+\cdots\right\}\times\left\{1+nx+\frac{n(n-1)}{\lfloor 2}x^2+\cdots\right\}$$

$$=1+(m+n)x+\frac{(m+n)(m+n-1)}{\lfloor 2}x^2+\quad\cdots\quad(4)$$

when m and n are positive integers.

Thus the identical relation (4) is established on the supposition that m and n are positive integers.

Now, when any algebraical expressions are multiplied together, the *form* of the resulting product is the same whether the letters involved represent whole numbers or fractions, positive or negative. This principle is known as the *Principle of the Permanence of Equivalent Forms*.

The equivalence of the two sides of the resulting identities (involved in this principle) evidently holds good when the number of terms on either side is finite. If the number of terms becomes infinite, however, then the equivalence holds good only when the series on either side are convergent.

In the present case, if m and n are not positive integers, the series involved in (4) are evidently infinite series, but they are all convergent if $x<1$ numerically [Art. 178]. Hence here the principle holds good, if $x<1$ numerically.

Hence if $x < 1$ numerically, using the above functional symbol, we must have $f(m) \times f(n) = f(m+n)$ for all values of m and n.

$\therefore \quad f(m) \times f(n) \times f(p) = f(m+n) \times f(p) = f(m+n+p).$

Proceeding thus, we have

$f(m) \times f(n) \times f(p) \times \cdots$ to r factors $= f(m+n+p+\ldots$ to r terms).

(i) Let $m = n = p = \ldots \frac{s}{r}$ where s and r are positive integers and $\frac{s}{r}$ is a fraction; then

$$f\left(\frac{s}{r}\right) \times f\left(\frac{s}{r}\right) \times f\left(\frac{s}{r}\right) \cdots \text{to } r \text{ factors}$$

$$= f\left(\frac{s}{r} + \frac{s}{r} + \frac{s}{r} + \cdots \text{to } r \text{ terms}\right)$$

i.e., $$\left\{f\left(\frac{s}{r}\right)\right\}^r = f(s) \; ;$$

but since s is a positive integer, $f(s) = (1+x)^s$.

$$\therefore \quad (1+x)^s = \left\{f\left(\frac{s}{r}\right)\right\}^r$$

$$\therefore \quad (1+x)^{\frac{s}{r}} = f\left(\frac{s}{r}\right)$$

$$= 1 + \frac{s}{r}x + \frac{\frac{s}{r}\left(\frac{s}{r} - 1\right)}{\lfloor 2}x^2 + \frac{\frac{s}{r}\left(\frac{s}{r} - 1\right)\left(\frac{s}{r} - 2\right)}{\lfloor 3}x^3 + \cdots\cdots$$

This proves the Binomial Theorem when the exponent is a *positive fraction.*

(ii) Again, since $f(m) \times f(n) = f(m+n)$ for all values of m and n, we have $f(-n) \times f(n) = f(-n+n) = f(0) = 1$.

$$\therefore \quad f(-n) = \frac{1}{f(n)} = \frac{1}{(1+x)^n} = (1+x)^{-n}$$

i.e., $$(1+x)^{-n} = f(-n)$$

$$= 1 + (-n)x + \frac{(-n)(-n-1)}{\lfloor 2}x^2 + \cdots\cdots$$

This proves the theorem when the exponent is a *negative quantity.*

Thus, when $x < 1$ numerically, the Binomial Theorem for any exponent is completely established. This proof is due to Euler.

NOTE. The Binomial Theorem is also valid when $x = 1$, provided $n > -1$; and valid when $x = -1$, provided n is positive. [For proof see C. Smith's *Algebra*, Art. 388.]

The Theorem is not valid when $x > 1$ numerically.

180. Points of Distinction.

The following points of distinction between Binomial expansions for a positive integral exponent and for any other exponent should be noticed :

(i) The number of terms is $n+1$ when n is a positive integer, but is *infinite* when n is not a positive integer.

(ii) The Binomial co-efficients can be written as ${}^nC_1$, ${}^nC_2$, ${}^nC_3, \cdots\cdots$ when n is a positive integer ; but they cannot be so written when n is not a positive integer.

(iii) The Binomial expansion is not convergent and hence is not valid when $x > 1$, if n is not a positive integer.

181. General Terms.

Since $\qquad (1+x)^n = 1 + nx + \dfrac{n(n-1)}{\underline{|2}}x^2 + \cdots\cdots$

t is obvious that the general term or the $(r+1)$th term

$$= \frac{n(n-1)(n-2)\cdots\cdots(n-r+1)}{\underline{|r}}x^r.$$

Similarly, the $(r+1)$th term in the expansion of $(1-x)^n$

$$= (-1)^r . \frac{n(n-1)(n-2)\cdots\cdots(n-r+1)}{\underline{|r}}x^r.$$

Again, $\qquad (1+x)^{-n} = 1 + (-n)x + \dfrac{-n(-n-1)}{\underline{|2}}x^2 + \ldots$

$$= 1 - nx + \frac{n(n+1)}{\underline{|2}}x^2 - \cdots\cdots$$

and the $(r+1)$th term $= \dfrac{-n(-n-1)(-n-2)\cdots(-n-r+1)}{\underline{|r}}x$

$$= (-1)^r . \frac{n(n+1)(n+2)\cdots(n+r-1)}{\underline{|r}}x^r.$$

Similarly, $(1-x)^{-n} = 1 + (-n)(-x) + \dfrac{-n(-n-1)}{\underline{|2}}(-x)^2 + \cdots$

$$= 1 + nx + \frac{n(n+1)}{\underline{|2}}x^2 + \cdots\cdots$$

d the $(r+1)$th term $= \dfrac{-n(-n-1)(-n-2)\cdots(-n-r+1)}{\underline{|r}}(-x)^r$

$$= (-1)^{2r} . \frac{n(n+1)(n+2)\cdots(n+r-1)}{\underline{|r}}x$$

$$= \frac{n(n+1)(n+2)\cdots(n+r-1)}{\underline{|r}}x^r.$$

182. A few Important Expansions.

The following expansions and their general or $(r+1)$th terms are important and should be remembered:

(i) $(1-x)^{-1}=1+x+x^2+x^3+\ldots\ldots$

The $(r+1)$th term $=x^r$.

(ii) $(1-x)^{-2}=1+2x+3x^2+4x^3+\ldots\ldots$

The $(r+1)$th term $=(r+1)x^r$.

(iii) $(1-x)^{-3}=1+3x+6x^2+10x^3+\cdots\cdots$

The $(r+1)$th term $=\dfrac{(r+1)(r+2)}{1.2}x^r$.

NOTE. That the Binomial Theorem does not hold good if $x>1$ is brought out very clearly by putting, say, $x=2$, in the above results, which become

$$-1=1+2+2^2+2^3+\ldots$$
$$1=1+2.2+3.2^2+4.2^3+\cdots$$
$$-1=1+3.2+6.2^2+10.2^3+\cdots$$

which are obviously absurd.

183. Expansion of $(a+x)^n$.

If n is not a positive integer, to expand $(a+x)^n$ we proceed in the following way:

(i) Let $a>x$, so that $\dfrac{x}{a}<1$.

Then, $(a+x)^n=\left\{a\left(1+\dfrac{x}{a}\right)\right\}^n=a^n\left(1+\dfrac{x}{a}\right)^n$

$$=a^n\left\{1+n.\frac{x}{a}+\frac{n(n-1)}{\underline{|2}}.\left(\frac{x}{a}\right)^2+\cdots\right\}\left[\because\frac{x}{a}<1\right]$$
$$=a^n+na^{n-1}x+\frac{n(n-1)}{\underline{|2}}a^{n-2}x^2+\ldots\ldots$$

(ii) Let $a<x$, so that $\dfrac{a}{x}<1$.

Then, $(a+x)^n=\left\{x\left(1+\dfrac{a}{x}\right)\right\}^n=x^n\left(1+\dfrac{a}{x}\right)^n$

$$=x^n\left\{1+n.\frac{a}{x}+\frac{n(n-1)}{\underline{|2}}.\left(\frac{a}{x}\right)^2+\cdots\right\}\left[\because\frac{a}{x}<1\right]$$
$$=x^n+nx^{n-1}a+\frac{n(n-1)}{\underline{|2}}.x^{n-2}a^2+\ldots\ldots$$

Similarly, if $x > 1$, $(1+x)^n$ is expanded in the following way :

$$(1+x)^n = \left\{x\left(1+\frac{1}{x}\right)\right\}^n = x^n\left(1+\frac{1}{x}\right)^n$$

$$= x^n\left\{1+n.\frac{1}{x}+\frac{n(n-1)}{\lfloor 2}.\left(\frac{1}{x}\right)^2+\cdots\right\}\left[\because \frac{1}{x} < 1\right]$$

$$= x^n + nx^{n-1} + \frac{n(n-1)}{\lfloor 2}x^{n-2} + \cdots\cdots$$

Example 1. *Expand* $(1+x)^{\frac{1}{2}}$ *to four terms, and find the general term.*

$$(1+x)^{\frac{1}{2}} = 1+\tfrac{1}{2}x+\frac{\frac{1}{2}(\frac{1}{2}-1)}{\lfloor 2}.x^2+\frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{\lfloor 3}x^3+\cdots$$

$$= 1+\tfrac{1}{2}x-\tfrac{1}{8}x^2+\tfrac{1}{16}x^3-\cdots\cdots$$

The $(r+1)$th term $= \dfrac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots\cdots(\frac{1}{2}-r+1)}{\lfloor r}\,x^r$

$$= \frac{1.(-1)(-3)(-5)\cdots\cdots(-2r+3)}{2^r\lfloor r}\,x^r$$

$$= (-1)^{r-1}.\frac{1.3.5\ldots(2r-3)}{2^r\lfloor r}x^r.$$

Example 2. *Expand* $(1-x)^{-1}$, *when* $x > 1$.

$$(1-x)^{-1} = \frac{1}{1-x} = -\frac{\frac{1}{x}}{1-\frac{1}{x}} = -\frac{1}{x}\left(1-\frac{1}{x}\right)^{-1}.$$

$$= -\frac{1}{x}\left(1+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\cdots\right)\left[\because \frac{1}{x} < 1\right]$$

$$= -\frac{1}{x}-\frac{1}{x^2}-\frac{1}{x^3}-\frac{1}{x^4}-\cdots\cdots$$

Example 3. *Expand* $\dfrac{a-x}{a+x}$ *in powers of* x, *when* $a > x$.

$$\frac{a-x}{a+x} = \frac{2a}{a+x}-1 = \frac{2}{1+\frac{x}{a}}-1 = 2\left(1+\frac{x}{a}\right)^{-1}-1$$

$$= 2\left\{1-\frac{x}{a}+\frac{x^2}{a^2}-\frac{x^3}{x^3}+\cdots\right\}-1\left[\because \frac{x}{a} < 1\right]$$

$$= 1-\frac{2x}{a}+\frac{2x^2}{a^2}-\frac{2x^3}{a^3}+\cdots$$

Example 4. *Expand* $(4+3x)^{\frac{5}{2}}$ *to four terms, and write down the general term.* [*Pun. I. E.* 1891]

$$(4+3x)^{\frac{5}{2}}=4^{\frac{5}{2}}(1+\tfrac{3}{4}x)^{\frac{5}{2}}$$

$$=4^{\frac{5}{2}}\left\{1+\tfrac{5}{2}\cdot\tfrac{3}{4}x+\frac{\frac{5}{2}(\frac{5}{2}-1)}{\lfloor 2}\cdot(\tfrac{3}{4}x)^2+\frac{\frac{5}{2}(\frac{5}{2}-1)(\frac{5}{2}-2)}{\lfloor 3}(\tfrac{3}{4}x)^3+\ldots\right\}$$

$$=32\left\{1+\tfrac{15}{8}x+\frac{5.3}{2.2.\lfloor 2}\cdot\tfrac{9}{16}x^2+\frac{5.3.1.}{2.2.2.\lfloor 3}\cdot\tfrac{27}{64}x^3+\cdots\right\}$$

$$=32+60x+\tfrac{135}{4}x^2+\tfrac{135}{32}x^3+\cdots$$

$$\text{The } (r+1)\text{th term}=4^{\frac{5}{2}}\cdot\frac{\frac{5}{2}(\frac{5}{2}-1)(\frac{5}{2}-2)(\frac{5}{2}-3)\cdots\cdots(\frac{5}{2}-r+1)}{\lfloor r}(\tfrac{3}{4}x)^r$$

$$=32.\frac{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.(-\frac{1}{2})(-\frac{3}{2})\cdots\cdots(\frac{5}{2}-r+1)}{\lfloor r}\cdot\frac{3^r}{4^r}.x^r$$

$$=(-1)^{r-3}\frac{2^5.5.3.1.1.3.\cdots\cdots(2r-7)}{2^r\lfloor r}\cdot\frac{3^r}{2^{2r}}.x^r$$

$$=(-1)^{r-3}\frac{5.3.1.1.3.\cdots\cdots(2r-7)}{2^{3r-5}\lfloor r}\cdot 3^r.x^r$$

Example 5. *Find the co-efficient of* x^r *in the expansion of*

$$(1+3x+6x^2+10x^3+\cdots ad\ inf.)^{\frac{4}{3}}.$$

We have $(1+3x+6x^2+10x^3+\cdots\cdots ad\ inf.)^{\frac{4}{3}}$

$$=\{(1-x)^{-3}\}^{\frac{4}{3}}=(1-x)^{-4}$$

$\therefore$ the co-efficient of x^r in the expansion of $(1-x)^{-4}$

$$=\frac{-4(-4-1)(-4-2)\cdots\cdots(-4-r+1)}{\lfloor r}\cdot(-1)^r$$

$$=(-1)^r.\frac{4.\ 5.\ 6.\ 7.\cdots(r+3)}{\lfloor r}(-1)^r$$

$$=(-1)^{2r}\frac{4.\ 5.\ 6.\ 7.\cdots r(r+1)(r+2)(r+3)}{1.2.3.4.5.\cdots r}$$

$$=\frac{(r+1)(r+2)(r+3)}{1.2.3}.$$

Example 6. *Find the first negative term in the expansion of* $(1+x)^{\frac{10}{3}}$, *x being positive.*

Since $T_{r+1}=\dfrac{n-r+1}{r}x\times T_r=\dfrac{3\frac{1}{3}-r+1}{r}.x\times T_r=\dfrac{4\frac{1}{3}-r}{r}.x\times T_r$,

and since x is positive, T_{r+1} will first be negative as soon as $4\frac{1}{3}-r$ is negative, *i.e.*, as soon as $r>4\frac{1}{3}$, *i.e.*, when $r=5$.

Hence the 6th term is the first negative term.

Example. 7. *Prove that*

$$\left(\frac{1+2x}{1+x}\right)^n=1+n\left(\frac{x}{1+2x}\right)+\frac{n(n+1)}{1.2}\left(\frac{x}{1+2x}\right)^2+\ldots\ldots$$

$$\left(\frac{1+2x}{1+x}\right)^n=\left(\frac{1+x}{1+2x}\right)^{-n}=\left(\frac{1+2x-x}{1+2x}\right)^{-n}=\left(1-\frac{x}{1+2x}\right)^{-n}$$

$$=1+n\left(\frac{x}{1+2x}\right)+\frac{n(n+1)}{1.2}\left(\frac{x}{1+2x}\right)^2+\ldots \quad \text{[Art. 181]}$$

EXERCISE 61.

Expand each of the following to four terms, and find the general term in each case :

1. $(1-x)^{-\frac{1}{2}}$. 2. $(1+x)^{\frac{5}{2}}$. 3. $(1+x)^{-\frac{2}{3}}$.

4. $(1-x)^{\frac{1}{3}}$. 5. $(1-3x)^{-\frac{2}{3}}$. 6. $(3+2x)^{\frac{1}{3}}$.

7. $(a+bx)^{-1}$. 8. $(a^2-x^2)^{-\frac{3}{4}}$. 9. $(1-4x)^{-\frac{1}{4}}$.

10. $\dfrac{1}{(a^{-\frac{1}{3}}+x^{-\frac{1}{3}})^3}$. 11. $\dfrac{1}{\sqrt[n]{1-x}}$. 12. $(1-2x)^{-\frac{3}{2}}$.

13. Write down the first five terms in the expansion of $(a^2+x)^{\frac{1}{2}}$. [Mad. F. A. 1891]

14. Write down the rth term in the expansion of $(1-x)^{\frac{3}{2}}$. [Cal. F. A. 1887]

15. Write down the rth term in the expansion of $(a-x)^{-\frac{1}{n}}$. [Cal. F. A. 1885]

16. Show that the co-efficient of x^r in the expansion of $(1+x)^{\frac{5}{3}}$ is $\frac{10}{3^r}\cdot\frac{1.4.7\cdots\cdots(3r-8)}{\lfloor r}(-1)^{r-2}$. [Cal. F. A. 1884]

17. Find the co-efficient of x^6 in the expansion of $(1+2x)^{\frac{5}{2}}$. [All. I. E. 1894]

18. Show that every term in the expansion of $(1-x)^{-\frac{p}{q}}$ is positive, and that the general term is

$$\frac{p(p+q)(p+2q)\cdots\cdots\{p+(r-1)q\}}{1.2.3.\cdots\cdots r.q^r}\cdot x^r.$$

19. Find the first negative term in the expansion of $(1+x)^{\frac{7}{3}}$.

20. Prove that every term in the expansion of $(1-x)^{\frac{1}{2}}$ after the first is negative.

21. If r be the greatest whole number contained in $\frac{p}{q}$, show that the first $r+2$ terms in the expansion of $(a+x)^{\frac{p}{q}}$ are positive, and that the $(r+3)$th term is the first negative term.

22. Prove that the co-efficient of x^m in the expansion of $(1-x)^{-(n+1)}$ is equal to the co-efficient of x^n in the expansion of $(1-x)^{-(m+1)}$.

23. If n is a positive integer, show that the co-efficient of x^r in the expansion of $(1-x)^{-n}$ is

$$\frac{(r+1)(r+2)(r+3)\cdots\cdots(r+n-1)}{\lfloor n-1}.$$

24. Expand $\dfrac{3a}{(a^3-x^2)^{\frac{1}{3}}}$ to 5 terms, and write down the $(r+1)$th term. [Cal. F. A. 1865]

25. Find the co-efficient of x^r in the expansion of

$$(1+2x+3x^2+4x^3+\cdots ad\ inf.)^2$$

26. Prove that

$$(1+x+x^2+x^3+\cdots ad\ inf.)(1+2x+3x^2+4x^3+\cdots ad\ inf.)$$
$$=\tfrac{1}{2}(1.2+2.3x+3.4x^2+4.5x^3\cdots ad\ inf.)$$ [Cal. F. A. 1877]

27. Show that the co-efficient of x^r in the expansion of

$$(1+2x+3x^2+4x^3+\cdots ad\ inf.)^{\frac{3}{4}}$$

is the same as the co-efficient of x^r in the expansion of

$$(1+3x+6x^2+10x^3+\cdots ad\ inf.)^{\frac{1}{2}}.$$

28. If $y=2x+3x^2+4x^3+\cdots$ to infinity, find x in a series of ascending powers of y.

29. Show that the $(r+1)$th term in the expansion of $(1-2x)^{-\frac{1}{2}}$ is $\dfrac{\lfloor 2r}{2^r(\lfloor r)^2}\cdot x^r$. [Mad. F. A. 1885]

30. Find the co-efficients of x^5 and x^6 in the expansion of

$$(1-2x+3x^2-4x^3+\cdots ad\ inf.)^{-\frac{5}{2}}.$$

31. Prove that $(1+x)^n=2^n\left\{1-n.\dfrac{1-x}{1+x}+\dfrac{n(n+1)}{1.2}\left(\dfrac{1-x}{1+x}\right)^2+\cdots\right\}$

32. Prove that $\left(\dfrac{1+x}{1-x}\right)^n=1+n.\left(\dfrac{2x}{1+x}\right)+\dfrac{n(n+1)}{1.2}.\left(\dfrac{2x}{1+x}\right)^2+\cdots$

[Cal. F. A. 1880]

33. Prove that

$$(a+x)^n = a^n\left\{1+n\left(\frac{x}{a+x}\right)+\frac{n(n+1)}{1.2}\left(\frac{x}{a+x}\right)^2+\ldots\ldots\right\}$$
$$= x^n\left\{1+n\left(\frac{a}{a+x}\right)+\frac{n(n+1)}{1.2}\left(\frac{a}{a+x}\right)^2+\ldots\ldots\right\}.$$

34. Prove that

$$1+n\frac{2n}{1+n}+\frac{n(n+1)}{1.2}\left(\frac{2n}{1+n}\right)^2+\ldots$$
$$=1+n.\frac{2n}{1-n}+\frac{n(n-1)}{1.2}\left(\frac{2n}{1-n}\right)^2+\ldots$$

35. Prove that

$$\left(\sqrt{\frac{1+x}{1-x}}\right)^n = 1+n.\left(\frac{x}{1+x}\right)+\frac{n(n+2)}{1.2}\cdot\left(\frac{x}{1+x}\right)^2$$
$$+\frac{n(n+2)(n+4)}{1.2.3}\left(\frac{x}{1+x}\right)^3+\ldots$$

184. Greatest Term.

To find the numerically greatest term in the expansion of $(1+x)^n$, when n is not a positive integer.

Since the terms in the expansion of $(1-x)^n$ are numerically the same as the corresponding terms in the expansion of $(1+x)^n$, it is obvious that the numerically greatest term of $(1-x)^n$ is the same as that of $(1+x)^n$. Hence it will be sufficient if we suppose x to be positive and proceed to find the greatest term of $(1+x)^n$.

We shall take $x < 1$ numerically, since the expansion is valid only in that case.

Let T_{r+1} and T_r denote respectively the $(r+1)$th and rth terms of $(1+x)^n$. Then we have

$$T_{r+1}=\frac{n-r+1}{r}\,x\times T_r=\left(\frac{n+1}{r}-1\right)x\times T_r.$$

Case I. Let n be a positive fraction.

As r increases, $\frac{n+1}{r}$ and therefore $\frac{n+1}{r}-1$ gradually diminishes and remains positive so long as $r<n+1$, and as soon as r becomes greater than $n+1$, $\frac{n+1}{r}-1$ becomes negative, but always remains numerically less than 1. Hence, since $x<1$, T_{r+1} is greatest when $\left(\frac{n+1}{r}-1\right)x$ is either just <1 or $=1$.

i.e., when $\frac{n+1}{r}$ is either just $>$ or $=1+\frac{1}{x}$,

i.e., " r " $<$ or $=\frac{n+1}{1+\frac{1}{x}}$,

i.e., " r " $<$ or $=\frac{(n+1)x}{1+x}$.

Now, if $\frac{(n+1)x}{1+x}=p$, an integer, then the $(p+1)$th term $=$the pth term, and both of them are the greatest terms.

If $\frac{(n+1)x}{1+x}$ be not an integer, let q be its integral part; then the $(q+1)$th term is the greatest.

Case II. Let n be negetive, either a fraction or an integer and equal to $-m$, where m is positive. Then

$$T_{r+1}=\frac{n-r+1}{r}\cdot x\times T_r=\frac{-m-r+1}{r}\cdot x\times T_r$$

$$=-\left(\frac{m-1}{r}+1\right)x\times T_r.$$

Hence, T_{r+1} is numerically greater than, equal to or less than T_r, according as the multiplying factor $\left(\frac{m-1}{r}+1\right)x$ $>$, $=$, or <1. Hence T_{r+1} is greatest,

when $\left(\frac{m-1}{r}+1\right)x$ is either just $>$ or $=1$,

i.e., " $\frac{m-1}{r}$ " $>$ or $=\frac{1}{x}-1$

i.e., " r " $<$ or $=\frac{m-1}{\frac{1}{x}-1}$

i.e., " r " $<$ or $=\frac{(m-1)x}{1-x}$.

Now, if $\frac{(m-1)x}{1-x}=p$, a positive integer, then the $(p+1)$th term $=$the pth term, and both of them are the greatest terms.

If $\frac{(m-1)x}{1-x}$ be positive but not an integer, let q be its integral part, then the $(q+1)$th term is the greatest.

If $\frac{(m-1)x}{1-x}$ be negative then, since $\frac{x}{1-x}$ is positive, $m-1$ must be negative, *i.e.*, $m<1$ (m being positive). Thus, $1-m$ is positive and <1, and therefore $\frac{1-m}{r}$ is positive and <1 (r being a positive integer). Hence, since $x<1$, the multiplying factor $\left(\frac{m-1}{r}+1\right)x$, *i.e.*, $\left(1-\frac{1-m}{r}\right)x$ is always less than 1, and each term is less than the preceding one. Hence in this case the first term, *i.e.*, unity, is the greatest.

The numerically greatest term in the expansion of $(a+x)^n$, when n is not a positive integer, can be determined similarly.

Example 1. *Find the numerically greatest term in the expansion of* $\left(1-\frac{x}{2}\right)^{\frac{5}{3}}$, *when* $x=\frac{4}{3}$.

The numerically greatest term of $\left(1-\frac{x}{2}\right)^{\frac{5}{3}}$ is the same as that of $\left(1+\frac{x}{2}\right)^{\frac{5}{3}}$. Since

$$T_{r+1}=\frac{\frac{5}{3}-r+1}{r}\cdot\frac{x}{2}\times T_r=\frac{\frac{5}{3}-r+1}{r}\cdot\frac{2}{3}\times T_r=\frac{16-6r}{9r}\times T_r.$$

$\therefore$ T_{r+1} is numerically greatest when $\frac{16-6r}{9r}$ is just $>$ or $=1$,

i.e., when 16 is just $>$ or $=15r$,

i.e., when r is just $<$ or $=1\frac{1}{15}$.

Now, r is a positive integer, therefore r must be equal to 1. Hence the 2nd term is the greatest.

Example 2 *Find the numerically greatest term in the expansion of* $(1-\frac{3}{4})^{-3}$.

The numerically greatest term of $(1-\frac{3}{4})^{-3}$ is the same as that of $(1+\frac{3}{4})^{-3}$

Since $T_{r+1}=\frac{-3-r+1}{r}\cdot\frac{3}{4}\times T_r=-\frac{2+r}{r}\cdot\frac{3}{4}\times T_r=-\frac{6+3r}{4r}\times T_r$,

$\therefore$ T_{r+1} is numerically greatest, when $\frac{6+3r}{4r}$ is just $>$ or $=1$,

i.e., when 6 is just $>$ or $=r$,

i.e., when r is just $<$ or $=6$.

Since r is a positive integer, therefore $r=6$.

Hence the 6th and 7th terms are the greatest.

EXERCISE 62.

Find the numerically greatest terms in the following expansions :

1. $(1+x)^{\frac{5}{2}}$, when $x=\frac{5}{6}$.
2. $(1+\frac{2}{3})^{\frac{11}{2}}$.
3. $(1+3x)^{\frac{7}{4}}$, when $x=\frac{2}{7}$.
4. $(1+\frac{1}{4})^{\frac{3}{5}}$.
5. $(1+\frac{2}{3})^{\frac{31}{5}}$. [Bom. P. E. 1888]
6. $(\sqrt{2}+1)^{\sqrt{2}}$.
7. $(1+x)^x$, when $x>1$.
8. $(1+x)^x$, when $x<1$.
9. $(1-x)^{-5}$, when $x=\frac{3}{4}$.
10. $(2+3x)^{-\frac{1}{2}}$, when $x=\frac{2}{5}$.

185. Application to Approximations.

If x is a small fraction (<1), so that x^2, x^3, x^4,......are rapidly diminishing, only a few terms of the series

$$1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3.}x^3+\cdot$$

need be taken as an approximation to the value of $(1+x)^n$. Thus, if x is so small that its square and higher powers may be neglected, then the approximate value of $(1+x)^n$ is $1+nx$. If the term containing x^2 be not neglected, a closer approximation to the value of $(1+x)^n$ is $1+nx+\frac{n(n-1)}{1.2}x^2$. Thus, the number of terms to be taken in determining the approximate value depends on the value of x, and on the degree of accuracy required.

Similarly, if x and y be both small fractions, we have

$$(1+x)^m(1+y)^n=(1+mx)(1+ny), \text{ approximately,}$$
$$=1+mx+ny, \text{ approximately,}$$

the product $mnxy$ which is small compared with x and y being neglected.

Example 1. *Calculate approximately the value of* $\sqrt[4]{623}$.

The complete fourth power nearest to 623 to 625, or 5^4 ; hence we proceed as follows :

$$\sqrt[4]{623}=\left(625-2\right)^{\frac{1}{4}}=625^{\frac{1}{4}}.\left(1-\frac{2}{625}\right)^{\frac{1}{4}}$$

$$=5\left\{1-\frac{1}{4}\cdot\frac{2}{625}+\frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{1.2}\cdot\left(\frac{2}{625}\right)^2-\cdots\right.$$

$$=5\{1-\tfrac{1}{4}(\cdot 0032)-\tfrac{3}{32}(\cdot 0032)^2-\}$$

$$=5\{1-\cdot 0008-\cdot 00000096\}(\text{approx.})=5(1-\cdot 00080096)$$

$$=4\cdot 9952952.$$

Example 2. *Find, by the Binomial Theorem, the square root of 2 to four places of decimals.*

$$\sqrt{2}=\sqrt{\left(\tfrac{49}{25}\times\tfrac{50}{49}\right)}=\tfrac{7}{5}\left(\tfrac{50}{49}\right)^{\frac{1}{2}}=\tfrac{7}{5}\left(\tfrac{49}{50}\right)^{-\frac{1}{2}}=\tfrac{7}{5}\left(1-\tfrac{1}{50}\right)^{-\frac{1}{2}}$$

$$=\tfrac{7}{5}\left\{1+\tfrac{1}{2}\cdot\tfrac{1}{50}+\frac{\frac{1}{2}\cdot\frac{3}{2}}{1.2}\left(\tfrac{1}{50}\right)^2+\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1.2.3.}\left(\tfrac{1}{50}\right)^3+\cdots\right\}$$

$$=\tfrac{7}{5}\left\{1+\tfrac{1}{100}+\tfrac{3}{20000}+\tfrac{1}{400000}+\cdots\right\}$$

$$=1{\cdot}4\times(1+{\cdot}01+{\cdot}00015+{\cdot}0000025+\cdots)=1{\cdot}4\times1{\cdot}0101525$$

$$=1{\cdot}4142, \text{ to four places of decimals,}$$

Example 3. *If x be so small that its square and higher powers may be neglected, prove that* $\dfrac{1+x}{1-x}=1+2x.$

$$\frac{1+x}{1-x}=(1+x)(1-x)^{-1}$$

$$=(1+x)(1+x) \quad \text{[neglecting squares and higher powers]}$$

$$=1+2x. \quad \text{[neglecting } x^2\text{]}$$

Example 4. *If p be nearly equal to q, show that*

$$\sqrt[3]{\frac{p}{q}}=\frac{2p+q}{p+2q}, \text{ approximately.}$$

Since p is nearly $=q$, $p-q$ is small compared with p or q,

$$\sqrt[3]{\frac{p}{q}}=\left\{\frac{(p+q)+(p-q)}{(p+q)-(p-q)}\right\}^{\frac{1}{3}}=\frac{(p+q)^{\frac{1}{3}}\left\{1+\dfrac{p-q}{p+q}\right\}^{\frac{1}{3}}}{(p+q)^{\frac{1}{3}}\left\{1-\dfrac{p-q}{p+q}\right\}^{\frac{1}{3}}}$$

$$=\frac{1+\frac{1}{3}\dfrac{p-q}{p+q}}{1-\frac{1}{3}\cdot\dfrac{p-q}{p+q}}=\frac{3(p+q)+(p-q)}{3(p+q)-(p-q)}=\frac{4p+2q}{2p+4q}=\frac{2p+q}{p+2q} \text{ (approx.)}$$

EXERCISE 63.

Find the value of the following to 5 places of decimals :

1. $\sqrt{99}$. 2. $\sqrt{24}$. [Cal. F. A. 1872]

3. $(626)^{\frac{3}{4}}$. 4. $\sqrt[3]{127}$. [Cal. F. A. 1875]

5. Find accurately to four places of decimals the value of $\dfrac{1}{\left(1-\frac{1}{10}\right)^{\frac{1}{10}}}$. [Cal. F. A. 1871]

6. Find the fourth root of 5 correct to 4 places of decimals.

If x is very small, prove approximately that

7. $\dfrac{(1+x)^{-\frac{1}{2}}(1-x)^{\frac{1}{2}}}{1+\frac{1}{2}x}=1-\frac{3}{2}x.$ 8. $\dfrac{(1-x)^{\frac{1}{2}}+(1+x)^{\frac{2}{3}}}{(1-x)+(1+x)^{\frac{1}{2}}}=1+x.$

9. $\dfrac{(1+\frac{3}{2}x)^{-7}+(1-2x)^{-1}}{(1+\frac{7}{2}x)^{-3}+(1+x)^{-2}}=1+2x.$

10. If c be a quantity so small that c^3 may be neglected in comparison with l^3, show that $\sqrt{\dfrac{l}{l+c}}+\sqrt{\dfrac{l}{l-c}}$ is very nearly equal to $2+\dfrac{3c^2}{4l^2}$. [Cal. F. A. 1888]

11. If p be nearly equal to q, then

$$\sqrt{\frac{p}{q}}=\frac{(n+1)p+(n-1)q}{(n-1)p+(n+1)q}\text{, nearly.}$$

12. Prove that if M differ from N^2 by a small quantity, the square root of M is approximately equal to

$$\frac{3}{2}N-\frac{(3N^2-M)^2}{8N^3}.$$ [Mad. F. A. 1891]

186. Application to Summation of Series.

The following examples will illustrate the method.

Example 1. *Find the sum of the series*

$$1+\frac{1}{4}+\frac{1.3}{4.8}+\frac{1.3.5}{4.8.12}+\frac{1.3.5.7}{4.8.12.16}+\cdots\textit{to infinity.}$$

The given series $=1+\frac{1}{2}.\frac{1}{2}+\dfrac{\frac{1}{2}\cdot\frac{3}{2}}{1.2}\cdot(\tfrac{1}{2})^2+\dfrac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{1.2.3}\cdot(\tfrac{1}{2})^3+\cdots$

$$=1+\tfrac{1}{2}.\tfrac{1}{2}+\frac{\frac{1}{2}(\frac{1}{2}+1)}{1.2}(\tfrac{1}{2})^2+\frac{\frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+2)}{1.2.3}(\tfrac{1}{2})^3+$$

$$=(1-\tfrac{1}{2})^{-\frac{1}{2}}=(\tfrac{1}{2})^{-\frac{1}{2}}=2^{\frac{1}{2}}=\sqrt{2}.$$

Example 2. *Find the sum of the series*

$$1+\frac{3}{3}+\frac{3.5}{3.6}+\frac{3.5.7}{3.6.9}+\cdots\cdots\textit{to infinity}\cdot$$

The given series $=1+\dfrac{3}{3}+\dfrac{\frac{3}{2}\cdot\frac{5}{2}}{1.2}(\tfrac{2}{3})^2+\dfrac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}{1.2.3}\cdot(\tfrac{2}{3})^3+\cdots$

$$=1+\tfrac{3}{2}.\tfrac{2}{3}+\frac{\frac{3}{2}(\frac{3}{2}+1)}{1.2}(\tfrac{2}{3})^2+\frac{\frac{3}{2}(\frac{3}{2}+1)(\frac{3}{2}+2)}{1.2.3}(\tfrac{2}{3})^3+.$$

$$=(1-\tfrac{2}{3})^{-\frac{3}{2}}=(\tfrac{1}{3})^{-\frac{3}{2}}=3^{\frac{3}{2}}=3\sqrt{3}.$$

Example 3. *Find the sum of the series*

$$1-\frac{2}{4}+\frac{2.5}{4.8}-\frac{2.5.8}{4.8.12}+\dots\textit{to infinity}.$$

The given series $=1-\dfrac{2}{4}+\dfrac{\frac{2}{3}\cdot\frac{5}{3}}{1.2}(\frac{3}{4})^2-\dfrac{\frac{2}{3}\cdot\frac{5}{3}\cdot\frac{8}{3}}{1.2.3}(\frac{3}{4})^3+\dots$

$$=1-\tfrac{2}{3}.\tfrac{3}{4}+\frac{\frac{2}{3}(\frac{2}{3}+1)}{1.2}(\tfrac{3}{4})^2-\frac{\frac{2}{3}(\frac{2}{3}+1)(\frac{2}{3}+2)}{1.2.3}(\tfrac{3}{4})^3+\dots$$

$$=(1+\tfrac{3}{4})^{-\frac{2}{3}}=(\tfrac{7}{4})^{-\frac{2}{3}}=(\tfrac{4}{7})^{\frac{2}{3}}=\sqrt[3]{(\tfrac{16}{49})}.$$

187. Sum of Co-efficients.

The following examples illustrate the method of finding out the sum of any number of co-efficients in the Binomial expansion for any exponent.

Example 1. *Find the sum of the first* $(r+1)$ *co-efficients in the expansion of* $(1-x)^{\frac{1}{2}}$.

Let $(1-x)^{\frac{1}{2}}=a_0+a_1x+a_2x^2+\dots\dots a_rx^r+\quad\dots\quad\dots\quad\dots$ (1)
where $a_0, a_1, a_2,\dots\dots a_r$ are the first $r+1$ co-efficients in the expansion.

Also $(1-x)^{-1}=1+x+x^2+x^3+\dots\dots+x^r+\quad\dots\quad\dots\quad\dots$ (2)

Hence the co-efficient of x^r in the product of the two series (1) and (2) must be equal to the co-efficient of x^r in the expansion of $(1-x)^{\frac{1}{2}}\times(1-x)^{-1}$, *i.e.*, in the expansion of $(1-x)^{-\frac{1}{2}}$.

But the co-efficient of x^r in the product of the two series is $a_0+a_1+a_2+\dots\dots+a_r$, which is the sum of the first $r+1$ co-efficients in the expansion of $(1-x)^{\frac{1}{2}}$.

$\therefore$ the required sum $=$ the co-efficient of x^r in $(1-x)^{-\frac{1}{2}}$

$$=\frac{\frac{1}{2}(\frac{1}{2}+1)(\frac{1}{2}+2)\dots\dots(\frac{1}{2}+r-1)}{\lfloor r}$$

$$=\frac{1.3.5\dots\dots(2r-1)}{2^r\lfloor r}.$$

Example 2. *Find the sum of n terms of the series*

$$1.2.3+2.3.4+3.4.5+\dots\dots$$

The $(r+1)$th term $=(r+1)(r+2)(r+3)=6\times\dfrac{(r+1)(r+2)(r+3)}{1.2.3}$

$=6\times$ the co-efficient of x^r in the expansion of $(1-x)^{-4}$

$\therefore$ the sum of the first n terms of the given series

$=6\times\{$the sum of the first n co-efficients in $(1-x)^{-4}\}$

$=6\times$ the co-efficient of x^{n-1} in the expansion of $(1-x)^{-5}$

[*Cf.* Example 1]

$$=6\times\frac{n(n+1)(n+2)(n+3)}{1.2.3.4}=\tfrac{1}{4}n(n+1)(n+2)(n+3).$$

EXERCISE 64.

Find the sum of the following infinite series :

1. $1+\frac{1}{4}+\frac{1.4}{4.8}+\frac{1.4.7}{4.8.12}+\ldots\ldots$

2. $1+2.\frac{1}{3^2}+\frac{2.5}{1.2}.\frac{1}{3^4}+\frac{2.5.8}{1.2.3}.\frac{1}{3^6}+\ldots$ [Cal. F. A. 1890]

3. $\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\frac{1.3.5.7}{3.6.9.12}+\ldots\ldots$ [Cal. F. A. 1895]

4. $1-\frac{1}{8}+\frac{1.3}{8.16}-\frac{1.3.5}{8.16.24}+\ldots\ldots$

5. $1+\frac{1}{4}+\frac{1.3}{4.6}+\frac{1.3.5}{4.6.8}+\ldots\ldots$

6. $1+2.\frac{1}{3}+3.\frac{1}{3^2}+4.\frac{1}{3^3}+\ldots\ldots$

7. $\frac{1}{2}+\frac{1.3}{2.4}.\frac{1}{4}+\frac{1.3.5}{2.4.6}.\frac{1}{4^2}+\ldots\ldots$

8. $\frac{1}{2.4}+\frac{1.3}{2.4.6}+\frac{1.3.5}{2.4.6.8}+\ldots\ldots$

9. Show that $1+\frac{1}{3}+\frac{1.3}{3.6}+\frac{1.3.5}{3.6.9}+\ldots=\sqrt{3}$. [Bom. P.E. 1889]

10. Show that $1+\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+\ldots=\sqrt{8}$. [Cal. F.A. 1894]

11. Show that $\sqrt{2}=\frac{7}{5}\left\{1+\frac{1}{10^2}+\frac{1.3}{1.2}.\frac{1}{10^4}+\frac{1.3.5}{1.2.3}.\frac{1}{10^6}+\cdots\right\}$.

[Mad. F. A. 1888]

12. Prove that $(1+x)^2=1+\frac{2x}{1+x}+\frac{3x^2}{(1+x)^2}+\frac{4x^3}{(1+x)^3}+\ldots\ldots$

13. Prove that, if n is a positive quantity,

$$n=1+\frac{n-1}{1.2}.\frac{1}{n}+\frac{(n-1)(2n-1)}{1.2.3}.\frac{1}{n^2}+\frac{(n-1)(2n-1)(3n-1)}{1.2.3.4}.\frac{1}{n^3}+\cdots$$

14. Find the sum of n terms of the series

$$1.2+2.3+3.4+4.5+\cdots$$

15. Find the sum of n terms of the series

$$1+n+\frac{n(n+1)}{1.2}+\frac{n(n+1)(n+2)}{1.2.3}+\cdots$$

188. Harder Expansions and Miscellanous Examples.

We work out below some harder expansions, and examples of miscellaneous types involving Binomial expansions.

Example 1. *Expand* $\left(\dfrac{a+x}{a-x}\right)^{\frac{1}{2}}$ *in ascending powers of* x.

$$\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}=\frac{(a+x)^{\frac{1}{2}}\cdot(a+x)^{\frac{1}{2}}}{(a-x)^{\frac{1}{2}}\cdot(a+x)^{\frac{1}{2}}}=\frac{a+x}{(a^2-x^2)^{\frac{1}{2}}}=\frac{a\left(1+\dfrac{x}{a}\right)}{a\left(1-\dfrac{x^2}{a^2}\right)^{\frac{1}{2}}}$$

$$=\left(1+\frac{x}{a}\right)\left(1-\frac{x^2}{a^2}\right)^{-\frac{1}{2}}$$

$$=\left(1+\frac{x}{a}\right)\left\{1+\frac{1}{2}\cdot\frac{x^2}{a^2}+\frac{1.3}{2.4}\cdot\frac{x^4}{a^4}+\frac{1.3.5}{2.4.6}\cdot\frac{x^6}{a^6}+\cdots\right\}$$

$$=1+\frac{x}{a}+\frac{1}{2}\cdot\frac{x^2}{a^2}+\frac{1}{2}\cdot\frac{x^3}{a^3}+\frac{1.3}{2.4}\cdot\frac{x^4}{a^4}+\frac{1.3}{2.4}\cdot\frac{x^5}{a^5}+\cdots\cdots$$

In this expansion, the co-eff. of $x^{2n}=\dfrac{1.3.5\cdots\cdots(2n-1)}{2.4.6\cdots\cdots 2n}\cdot\dfrac{1}{a^{2n}}$,

and the co-eff. of $x^{2n+1}=\dfrac{1.3.5\cdots\cdots(2n-1)}{2.4.6\cdots\cdots 2n}\cdot\dfrac{1}{a^{2n+1}}$.

Example 2. *Find the co-efficient of* x^n *in the expansion of* $\dfrac{(1+x)^2}{(1-x)^3}$.

$$\frac{(1+x)^2}{(1-x)^3}=(1+2x+x^2)(1-x)^{-3}$$
$$=(1+2x+x^2)\{1+3x+6x^2+\cdots+\tfrac{1}{2}(n+1)(n+2)x^n+\cdots\}$$

$\therefore$ the co-efficient of $x^n=\frac{1}{2}(n+1)(n+2)+2.\frac{1}{2}n(n+1)+\frac{1}{2}(n-1)n$
$=\frac{1}{2}\{(n^2+3n+2)+(2n^2+2n)+(n^2-n)\}$
$=2n^2+2n+1$.

Example 3. *If* $p_r=\dfrac{3.7.11\cdots(4r-1)}{4.8.12\cdots\cdots 4r}$ *and* $q_r=\dfrac{1.5.9\cdots\cdots(4r-3)}{4.8.12\cdots\cdots 4r}$, *prove that* $p_r+p_{r-1}q_1+p_{r-2}q_2+\cdots\cdots+p_1q_{r-1}+q_r=1$.

We have $p_r=\dfrac{3.7.11\cdots\cdots(4r-1)}{4.8.12\cdots\cdots 4r}=\dfrac{\frac{3}{4}.\frac{7}{4}.\frac{11}{4}\cdots\cdots\frac{4r-1}{4}}{1.2.3\cdots\cdots r}$

$$=\frac{\frac{3}{4}(\frac{3}{4}+1)(\frac{3}{4}+2)\cdots\cdots(\frac{3}{4}+r-1)}{1.2.3\cdots\cdots r}$$

$=$ the co-eff. of x^r in the expansion of $(1-x)^{-\frac{3}{4}}$.

$$\therefore\ (1-x)^{-\frac{3}{4}}=1+p_1x+p_2x^2+p_3x^3+\cdots+p_{r-1}x^{r-1}+p_rx^r+\cdots \quad (1)$$

Similarly, $q_r = \frac{1.5.9\cdots\cdots(4r-3)}{4.8.12\cdots4r} = \frac{\frac{1}{4}.\frac{5}{4}.\frac{9}{4}\cdots\cdots\frac{4r-3}{4}}{1.2.3\cdots\cdots r}$

$$= \frac{\frac{1}{4}(\frac{1}{4}+1)(\frac{1}{4}+2)\cdots\cdots(\frac{1}{4}+r-1)}{1.2.3\cdots\cdots r}$$

= the co-eff. of x^r in the expansion of $(1-x)^{-\frac{1}{4}}$.

$\therefore \quad (1-x)^{-\frac{1}{4}} = 1 + q_1x + q_2x^2 + q_3x^3 + \ldots + q_{r-1}x^{r-1} + q_rx^r + \cdots (2)$

$\therefore$ from (1) and (2) we have

$$(1+p_1x+p_2x^2+\cdots\cdots+p_rx^r+\ldots)(1+q_1x+q_2x^2+\cdots\cdots+q_rx^r+\cdots$$

$$=(1-x)^{-\frac{3}{4}} \times (1-x)^{-\frac{1}{4}} = (1-x)^{-1}$$

$$= 1 + x + x^2 + x^3 + \ldots + x^r + \cdots\cdots$$

Hence, equating the co-efficient of x^r on both sides,

$$p_r + p_{r-1}q_1 + p_{r-2}q_2 + \cdots\cdots + p_1q_{r-1} + q_r = 1.$$

Example 4. *Prove that*

$$2^{n-1} - (n-2).2^{n-3} + \frac{(n-3)(n-4)}{1.2}.2^{n-5} - \frac{(n-4)(n-5)(n-6)}{1.2.3}2^{n-7} + \ldots$$

$= n$, *n being any positive integer.*

We have $(1-x)^{-2} = \{(1-x)^2\}^{-1} = (1-2x+x^2)^{-1} = \{1-x(2-x)\}^{-1}$.

Expanding both sides,

$$1 + 2x + 3x^2 + 4x^3 + \cdots\cdots + nx^{n-1} + \cdots\cdots$$

$$= 1 + x(2-x) + x^2(2-x)^2 + \ldots + x^{n-2}(2-x)^{n-2} + x^{n-1}(2-x)^{n-1} + \cdots$$

This being an identity, the co-efficient of x^{n-1} on both sides must be equal.

Now, the co-eff. of x^{n-1} on the left-hand side $= n$.

The co-eff. of x^{n-1} on the right-hand side is obtained thus:

the co-eff. of x^{n-1} in $x^{n-1}(2-x)^{n-1} = 2^{n-1}$;

" " $x^{n-2}(2-x)^{n-2} = -(n-2).2^{n-3}$;

" " $x^{n-3}(2-x)^{n-3} = \frac{(n-3)(n-4)}{1.2}2^{n-5}$;

and so on. Hence, collecting these together,

the co-eff. of $x^{n-1} = 2^{n-1} - (n-2).2^{n-3}$

$$+ \frac{(n-3)(n-4)}{1.2}.2^{n-5} - \ldots$$

Hence the required identity is established.

EXERCISE 65.

1. Write down the co-efficient of x^{10} in $\dfrac{1+x}{(1-x)^3}$.

[Cal. Int. 1937]

2. Find the co-efficient of x^n in the expansion of $\dfrac{(1+x)^2}{(1-x)^2}$.

[Cal. Int. 1939]

3. Find the co-efficient of x^r in the expansion of

$$\frac{1+4x^2+x^4}{(1-x)^4}.$$ [Cal. F. A. 1881]

4. Find the co-efficient of x^n in the expansion of

$$\frac{(1+x)^2}{(1-2x)^3}.$$ [Mad. F.A. 1887]

5. Show that the co-efficient of x^n in the expansion of $\dfrac{(1+x)^n}{1-x}$ is 2^n.

6. Find the first three terms in the expansion of

$$\frac{1}{(1+x)^2\sqrt{(1+4x)}}$$ [Cal. F. A. 1901]

7. Show that, if a and x be each < 1,

$$\frac{1}{(1-x)(1+ax)}=1+(1+a)x+(1+a+a^2)x^2+(1+a+a^2+a^3)x^3+\cdots$$

8. Prove that the co-efficient of x^n in the expansion of $(1-9x+20x^2)^{-1}$ is $5^{n+1}-4^{n+1}$. [Bom. P. E. 1893].

9. Show that the co-efficient of x^r in the expansion of $\dfrac{1}{(1-x)(1-2x)(1-3x)}$ in ascending powers of x is $\frac{1}{2}(3^{n+2}-2^{n+3}+1)$.

10. Show that the co-efficient of x^n in the expansion of $\dfrac{2-3x^2}{1-3x+2x^2}$ is 2^n+1.

11. Find the $(r+1)$th term in the expansion of $\dfrac{x}{(1-ax)(1-bx)}$, supposing x less than both $\dfrac{1}{a}$ and $\dfrac{1}{b}$.

12. Prove that the co-efficient of x^n in the expansion of $\dfrac{1}{1+x+x^2}$ is $1, 0, -1$ according as n is of the form $3m$, $3m-1$, or $3m+1$.

13. Prove that the co-efficient of x^n in the expansion of $\dfrac{1}{1+x+x^2+x^3}$ is 1 or -1, according as n is of the form $4m$ or $4m+1$, and in all other cases it is zero.

14. If $p_r=\frac{1.3.5\cdots(2r-1)}{2.4.6\ldots 2r}$ and $q_r=\frac{5.7.9\cdots(2r+3)}{2.4.6\cdots 2r}$, prove that

(i) $p_r+p_{r-1}q_1+p_{r-2}q_2+\cdots+p_1q_{r-1}+q_r=\frac{1}{2}(r+1)(r+2)$.

(ii) $p_r+p_{r-1}p_1+p_{r-2}p_2+\cdots+p_1p_{r-1}+p_r=1$.

*15. Prove that the sum of the middle terms in the expansion of $(1+x)^n$, for all even values of n including zero, is $(1-4x)^{-\frac{1}{2}}$. $\longrightarrow$ 1952.

16. Show that

$$\frac{1+11x+11x^2+x^3}{(1-x)^5}=1+2^4x+3^4x^2+4^4x^3+\cdots+n^4x^{n-1}+\cdots$$

17. Prove that $2^n=1+\frac{(n+1)n}{\lfloor\underline{2}}+\frac{(n+1)n(n-1)(n-2)}{\lfloor\underline{4}}+\cdots$

18. If n be any positive integer, show that

$$1-\frac{n^2}{1^2}+\frac{n^2(n^2-1^2)}{1^2.2^2}-\frac{n^2(n^2-1^2)(n^2-2^2)}{1^2.2^2.3^2}+\ldots=0.$$

19. Prove that the sum of n terms of the series

$$1-\frac{n-1}{1.2}+\frac{(n-1)(n-2)}{1.2.3}-\frac{(n-1)(n-2)(n-3)}{1.2.3.4}+\cdots=\frac{1}{n}.$$

20. Prove that $\left(\frac{1}{x}-\frac{1}{a}\right)^{-n}=x^n+n.\frac{x^{n+1}}{a}+\frac{n(n+1)}{1.2}.\frac{x^{n+2}}{a^2}+\cdots$

21. If n be any positive integer prove that

$$3^{n-1}-\frac{n-2}{1}.3^{n-3}+\frac{(n-3)(n-4)}{1.2}.3^{n-5}-\frac{(n-4)(n-5)(n-6)}{1.2.3}.3^{n-7}+\ldots$$

$$=n\left\{1+\frac{n^2-1}{\lfloor\underline{3}}+\frac{(n^2-1)(n^2-4)}{\lfloor\underline{5}}+\frac{(n^2-1)(n^2-4)(n^2-9)}{\lfloor\underline{7}}+\cdots\right\}.$$

22. Show that, if n is a positive integer,

$$1-\frac{n^2-1}{\lfloor\underline{3}}.2+\frac{(n^2-1)(n^2-4)}{\lfloor\underline{5}}.2^2-\frac{(n^2-1)(n^2-4)(n^2-9)}{\lfloor\underline{7}}.2^3+\cdots$$

$=0$ when n is even, and $=(-1)^{\frac{n-1}{2}}\frac{1}{n}$ when n is odd.

23. If n is any positive integer, show that

$$1-(n-1)+\frac{(n-2)(n-3)}{\lfloor\underline{2}}-\frac{(n-3)(n-4)(n-5)}{\lfloor\underline{3}}+\cdots=\pm1 \text{ or } 0.$$

24. If $x<1$, show that

$$\frac{x}{1-x^2}-\frac{x^3}{1-x^6}+\frac{x^5}{1-x^{10}}-\ldots\ldots=\frac{x}{1+x^2}+\frac{x^3}{1+x^6}+\frac{x^5}{1+x^{10}}+\ldots\ldots$$

25. If $(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_nx^n$, prove that

$$C_1-\frac{1}{2}C_2+\frac{1}{3}C_3-\frac{1}{4}C_4+\cdots+(-1)^{n-1}\frac{1}{n}C_n=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}.$$

CHAPTER XVI

LOGARITHMS

189. Definitions.

Consider the relation $5^3 = 125$.

There are three quantities involved here : the number 5 (called the *base*), the number 3 (called the *power*), and the resulting number 125. If any two of these quantities are given, the third should be found out. Three questions, accordingly, can be asked regarding a relation like this

First we may ask ; If 5 is raised to the power 3, what is the result ? The answer is 125, found by multiplication.

Secondly we may ask : What is the number which being raised to the power 3 becomes 125 ? The answer is 5, found by extracting the cube root.

Thirdly we may ask : What is the power to which 5 must be raised to produce 125 ? The answer, as we see in this particular case, is 3 ; but it cannot be found out by any of the processes explained before.

The answer to this question involves the introduction of a new kind of function, called *logarithm*, which may be defined thus :

The logarithm of a number with respect to a given base is the index of the power to which the base must be raised to produce that number.

Thus, in the particular case considered above, 3 is the logarithm of 125 with respect to 5 as base.

Generally, if $a^x = n$, then x is the logarithm of n with respect to the base a ; and this is shortly written thus :

$$x = \log_a n.$$

Conversely, if $x = \log_a n$, then $a^x = n$.

Thus, since $5^3 = 125$, we have $3 = \log_5 125$;

since $10^4 = 10000$, we have $4 = \log_{10} 10000$;

since $2^{-5} = \frac{1}{32}$, we have $-5 = \log_2(\frac{1}{32})$;

since $64^{\frac{1}{6}} = 2$, we have $\frac{1}{6} = \log_{64} 2$;

and so on.

Any number (other than unity) may be chosen as the base, and logarithms of other numbers found with respect to it. But since a positive number raised to any real power, positive or negative, always produces a positive result, it follows that *the logarithm of a nagative number with respect to a positive base cannot be a real quantity, and is in fact imaginary.*

Further, *unity cannot be chosen as the base*, for 1 raised to any finite power produces always 1, and no other number.

Usually, the number which is chosen as the base is 10, the *radix* of the decimal system of notation, and the system of logarithms with respect to 10 as base is known as the *Common system.*

There is another system of logarithms based upon another number, denoted by the letter e, which will be explained in the next chapter, and which system is very useful in theoretical investigations. This is known as the *Napierian* system ; it has been so named after John Napier, the inventor of logarithms.

190. Fundamental Properties of Logarithms.

(*i*) Since $a^0=1$, we have $\log_a 1=0$,
i. e., the logarithm of unity to any finite base is zero.

(*ii*) Since $a^1=a$, we have $\log_a a=1$,
i.e., the logarithm of the base itself is unity.

(*iii*) Since $a^{-1}=\frac{1}{a}$, we have $\log_a\left(\frac{1}{a}\right)=-1$,
i.e., the logarithm of the reciprocal of the base is -1.

(*iv*) Since $a^{\infty}=0$ if $a<1$, and $a^{-\infty}=0$ if $a>1$, we have $\log_a 0=\infty$ *or* $-\infty$, *according as* $a<$ or >1.

(*v*) Since $a^{\infty}=\infty$ if $a>1$, and $a^{-\infty}=\infty$ if $a<1$, we have $\log_a\infty=\infty$ *or* $-\infty$, *according as* $a>$ or <1.

(*vi*) If $a^y=x$, we have from the definition $y=\log_a x$.

Hence, substituting the value of y, we have $a^{\log_a x}=x$.

191. General Propositions on Logarithms.

(*i*) *To prove that* $\log_a(mn)=\log_a m+\log_a n$.

Let $\log_a m=x$, and $\log_a n=y$,

then, $m=a^x$ and $n=a^y$;

$\therefore$ $mn=a^x\times a^y=a^{x+y}$;

$\therefore$ $\log_a(mn)=x+y=\log_a m+\log_a n$.

(*ii*) *To prove that* $\log_a(mnp\cdots) = \log_a m + \log_a n + \log_a p + \cdots$

Let, $\log_a m = x$, $\log_a n = y$, $\log_a p = z \cdots\cdots$

Then $\quad m = a^x, \quad n = a^y, \quad p = a^z, \cdots\cdots$

$\therefore \quad mnp\ldots\ldots = a^x \times a^y \times a^z \times \ldots\ldots = a^{x+y+z+}\ldots\ldots$

$\therefore \quad \log_a(mnp\ldots) = x + y + z + \ldots = \log_a m + \log_a n + \log_a p + \ldots$

(*iii*) *To prove that* $\log_a(m^n) = n\log_a m$.

Let $\log_a m = x$, then $a^x = m$, and hence $m^n = (a^x)^n = a^{nx}$

$$\therefore \quad \log_a(m^n) = nx = n \log_a m.$$

(*iv*) *To prove that* $\log\left(\dfrac{m}{n}\right) = \log_a m - \log_a n$.

Let $\log_a m = x$, and $\log_a n = y$; then $m = a^x$ and $n = a^y$.

$$\frac{m}{n} = a^x \div a^y = a^{x-y}.$$

$$\log_a\left(\frac{m}{n}\right) = x - y = \log_a m - \log_a n.$$

(*v*) *To prove that* $\log_b a = \dfrac{1}{\log_a b}$.

Let $\log_a b = x$, then $a^x = b$, and hence $a = b^{\frac{1}{x}}$.

$$\therefore \quad \log_b a = \frac{1}{x} = \frac{1}{\log_a b}.$$

It appears from the above propositions that by the introduction of logarithms the process of multiplication is reduced to that of addition, division to subtraction, involution or evolution (*i.e.*, raising to powers and extracting roots) to multiplication. Therein lies the practical utility of logarithms.

Cor. *To prove* $\log_{a^n} m = \dfrac{1}{n} \log_a m$.

Let $\quad \log_a m = x$, then $m = a^x = (a^n)^{\frac{x}{n}}$

$$\therefore \quad \log_{a^n} m = \frac{x}{n} = \frac{1}{n} \log_a m.$$

192. Logarithms to different bases.

To prove that $\log_a m = \log_b m \times \log_a b.$

Let $\log_a m = x$, then $a^x = m.$

Let $\log_b m = y$, then $b^y = m.$

Hence, $a^x = b^y$, or $b = a^{\frac{x}{y}}$

$\therefore \quad \log_a b = \dfrac{x}{y} = \dfrac{\log_a m}{\log_b m}.$

Hence, $\log_a m = \log_b m \times \log_a b.$

This relation can also be written thus :

$$\log_b m = \frac{\log_a m}{\log_a b}.$$

i.e., the logarithm of a number to a new base is equal to its logarithm to the old base divided by the logarithm of the new base to the old.

Thus, since $\log_5 625 = 4$, we have

$$\log_{25} 625 = \frac{\log_5 625}{\log_5 25} = \frac{4}{2} = 2.$$

Cor. To prove that $\log_b a \times \log_c b \times \log_a c = 1.$

From the above Theorem, we have

$\log_a b \div \log_a c = \log_c b.$

$\therefore \quad \log_c b \times \log_a c = \log_a b = \dfrac{1}{\log_b a}$ [Art. 191(v)]

Hence, $\log_b a \times \log_c b \times \log_a c = 1.$

Example 1. *Find the logarithm of* 1600 *to the base* $2.\sqrt[3]{5}$.

Let x be the required logarithm.

Then from the definition, we have $(2.\sqrt[3]{5})^x = 1600.$

$\therefore \quad (8 \times 5)^{\frac{x}{3}} = 1600$; or, $(40)^{\frac{x}{3}} = 40^2.$

$\therefore \quad \frac{1}{3}x = 2$; or, $x = 6.$

Hence the required logarithm is 6.

Example 2. *Simplify* $\log \sqrt{\frac{15}{14}} + \log \sqrt{\frac{21}{20}}$.

The given expression $= \log \{\sqrt{\frac{15}{14}} \times \sqrt{\frac{21}{20}}\} = \log \sqrt{\{\frac{15}{14} \times \frac{21}{20}\}}$

$$= \log (\tfrac{9}{8})^{\frac{1}{2}} = \log \frac{3}{2\sqrt{2}}$$

$$= \log 3 - \log 2\sqrt{2}$$

$$= \log 3 - \log (2^{\frac{3}{2}})$$

$$= \log 3 - \tfrac{3}{2} \log 2.$$

Example 3. *If* $\frac{\log x}{y-z}=\frac{\log y}{z-x}=\frac{\log z}{x-y}$ *prove that* $x^x y^y z^z=1$.

Let each of the given ratios be equal to k; then

$$k=\frac{x\log x}{x(y-z)}=\frac{y\log y}{y(z-x)}=\frac{z\log z}{z(x-y)}$$
$$=\frac{x\log x+y\log y+z\log z}{x(y-z)+y(z-x)+z(x-y)}$$
$$=\frac{\log(x^x y^y z^z)}{0}$$

$\log(x^x y^y z^z)=0$

$\therefore\quad x^x y^y z^z=1$ $\quad[\because \log 1=0.]$

Example 4. *Prove that* $7\log\frac{10}{9}-2\log\frac{25}{24}+3\log\frac{81}{80}=\log 2$.
[*Cal. Int.* 1923]

We have $7\log\frac{10}{9}=7(\log 10-\log 9)$
$=7(\log 2+\log 5-2\log 3)$
$=7\log 2+7\log 5-14\log 3$;

$2\log\frac{25}{24}=2(\log 25-\log 24)$
$=2[\log 5^2-\log(2^3\times 3)]$
$=2(2\log 5-3\log 2-\log 3)$
$=4\log 5-6\log 2-2\log 3$;

$3\log\frac{81}{80}=3(\log 81-\log 80)$
$=3[\log 3^4-\log(2^4\times 5)]$
$=3[4\log 3-4\log 2-\log 5]$
$=12\log 3-12\log 2-3\log 5$.

Hence the given expression
$=(7+6-12)\log 2+(-14+2+12)\log 3+(7-4-3)\log 5$
$=\log 2$.

EXERCISE 66.

1. Find the logarithms of :

(i) 2025 to the base $3\sqrt{5}$. (ii) $\sqrt[3]{25}$ to the base $\sqrt{5}$.

(iii) ·03125 to the base ·125. (iv) 1728 to the base $\frac{1}{6}\sqrt{3}$.

(v) 16 to the base 8. (vi) 144 to the base $2\sqrt{3}$.

[Cal. F. A. 1873] [Cal. F. A. 1876]

2. Simplify the following :

(i) $\log \frac{9}{14} + \log \frac{35}{24} - \log \frac{15}{16}$.

(ii) $2 \log \frac{25}{24} + 3 \log \frac{9}{8} - 4 \log \frac{3}{4}$.

(iii) $\dfrac{\log \frac{9}{8} + \log \frac{35}{33} - \log \frac{7}{11}}{\log \frac{5}{3} - \log \frac{\;}{18} + \log \frac{7}{15}}$.

3. Show that $7 \log \frac{16}{15} + 5 \log \frac{25}{24} + 3 \log \frac{81}{80} = \log 2$.

[Cal. Int. 1936]

4. Show that
$\log_{10} 2 + 16 \log_{10} \frac{16}{15} + 12 \log_{10} \frac{25}{24} + 7 \log_{10} \frac{81}{80} = 1$.

[Cal. Int. 1940]

5. The logarithm of a number to the base $\sqrt{2}$ is k. What is its logarithm to the base $2\sqrt{2}$?

6. If a, b, c, d are any four positive numbers, prove that
$\log_b a \times \log_c b \times \log_d c = \log_d a$. [Cal. Int. 1942]

7. If $a, b, c, \ldots$ be in G. P., show that

(i) $\log a, \log b, \log c, \ldots$ are in A. P.

(ii) $\log_a n, \log_b n, \log_c n, \cdots$ are in H. P.

8. Show that $(\log a)^2 - (\log b)^2 = \log (ab) \log \left(\dfrac{a}{b}\right)$.

9. If $a^2 + b^2 = 7ab$, show that $\log \{\frac{1}{3}(a+b)\} = \frac{1}{2}(\log a + \log b)$.

10. Show that $\log_{b^n} a^n = \log_b a$.

11. Show that (i) $x = a^{\log_a (a^{\log_a x})}$ (ii) $a^x = b^{x \log_b a}$.

12. Prove that $\log_{b^2} a + \log_{c^2} b \times \log_{a^2} c = \frac{1}{8}$.

13. Show that
$\log_a x . \log_b y . \log_c z = \log_b x . \log_c y . \log_a z = \log_c x . \log_a y . \log_b z$.

14. If $\dfrac{\log x}{b-c} = \dfrac{\log y}{c-a} = \dfrac{\log z}{a-b}$ prove that

(i) $x^a y^b z^c = 1$. (ii) $x^{b+c} y^{c+a} z^{a+b} = 1$.

15. If $x = \log_a bc$, $y = \log_b ca$, $z = \log_c ab$, prove that
$xyz = x + y + z + 2$.

16. If $xy^{p-1} = a$, $xy^{q-1} = b$, $xy^{r-1} = c$ prove that
$(q-r) \log a + (r-p) \log b + (p-q) \log c = 0$.

17. If $x = \log_c b + \log_b c$, $y = \log_a c + \log_c a$, $z = \log_b a + \log_a b$, prove that $x^2 + y^2 + z^2 - xyz = 4$.

18. Prove that $x^{\log y - \log z} \times y^{\log z - \log x} \times z^{\log x - \log y} = 1$.

[Cal. Int. 1939 & 1944.]

193. Common System of Logarithms.

As has already been stated, the base usually chosen for purposes of computation is 10; and the system of logarithms based upon 10 is called the *Common System*. This system was first introduced in 1615 by Briggs, a contemporary of John Napier, the inventor of logarithms, and is hence sometimes called the *Briggsian system*.

For convenience of calculation, tables of logarithms of all natural numbers from 1 to 108000 to the base 10 have been constructed, giving logarithms to seven places of decimals. In all practical applications, the student will have to refer to the tables.

The articles that follow relate to the properties of logarithms of the Common system. The base 10 is usually not written but is understood.

194. Characteristic and Mantissa.

The integral part of a logarithm is called the *characteristic*, and the decimal part is called the *mantissa*.

For instance, since $\log_{10}450 = 2{\cdot}6532126$, the characteristic of $\log_{10}450$ is 2 and its mantissa is $\cdot 6532126$.

It should be noted that in the common system of logarithms as tabulated, the characteristic of a logarithm may be either positive or negative, but *the mantissa is always retained as positive*. Thus, the positive fractional part is called the mantissa, and the integral part obtained after expressing the mantissa positively is called the characteristic of the logarithm.

For instance, we have $\quad \log_{10}2 = \cdot 3010300$.

$$\therefore \log_{10}200 = \log_{10}(2 \times 100) = \log_{10}2 + \log_{10}100$$
$$= \cdot 3010300 + 2 = 2{\cdot}3010300.$$

Here 2 is the characteristic and $\cdot 3010300$ is the mantissa.

$$\text{Again } \log_{10}\cdot 02 = \log_{10}\tfrac{2}{100} = \log_{10}2 - \log_{10}100$$
$$= \cdot 3010300 - 3 = -3 + \cdot 301030.$$

Here -3 is the characteristic and $\cdot 3010300$ is the mantissa. It should be carefully noted that, $-3 + \cdot 3010300 = -2{\cdot}6989700$, but $\cdot 6989700$ is not the mantissa. for in $-2{\cdot}6989700 = (-2 - \cdot 6989700)$ the decimal part is negative, but the mantissa is the *positive* fractional part. And so also -2 is not the characteristic. Again since $-3{\cdot}3010300$ indicates in Arithmetical notation that both the integral and the fractional

parts are negative, the number $-3+\cdot3010300$ is written as $\bar{3}\cdot3010300$ (*i.e.*, with a bar on the integral part only), showing thereby that the characteristic alone is negative, and the mantissa positive.

195. Logarithms of numbers having the same significant digits.

To prove that the logarithms of numbers having the same significant digits have the same mantissa and differ only in their characteristics.

Let M and N be two numbers having the same significant digits. Then it is obvious that $M=N\times10^n$, where n is an integer, either positive or negative.

$$\therefore \quad \log_{10}M=\log_{10}N+\log_{10}10^n$$
$$=\log_{10}N+n.$$

Thus, the logarithms of M and N to the base 10 differ only by an integer n, either positive or negative, and therefore they have the same mantissa.

For instance, the numbers 4037, 4·037, 40·37, ·004037, 403700,⋯ which are such that any one of them may be obtained from any other by multiplying or dividing by an integral power of 10, and hence have the same significant digits, have the same mantissa.

196. Characteristics found by Inspection.

To find the characteristic of the logarithm of any number.

(*i*) Let the number be *greater than unity.*

A number containing one digit in the integral part lies between 1 and 10, and therefore its logarithm lies between 0 and 1.

Thus there is no integral part in the logarithm.

Hence the characteristic of the logarithm of a number having one digit only in the integral part is 0.

A number containing two digits in the integral part lies between 10 and 10^2, and therefore its logarithm lies between 1 and 2. Hence the characteristic of the logarithm of a number having 2 digits in its integral part is 1.

A number containing three digits in the integral part lies between 10^2 and 10^3, and therefore its logarithm lies between 2 and 3 Hence the characteristic of the logarithm of a number having 3 digits in its integral part is 2.

Hence, generally, a number containing n digits in the integral part lies between 10^{n-1} and 10^n, and therefore its logarithm lies between $n-1$ and n. Therefore the characteristic of the logarithm is $n-1$.

(ii) Let the number be *less than unity*.

If there is no zero between the decimal point and the first significant figure, the number lies between 10^{-1} and 1, and hence its logarithm lies between -1 and 0. Since the mantissa, that is, the decimal part must be positive, the characteristic is -1.

If there is one zero between the decimal point and the first significant figure, the number lies between 10^{-2} and 10^{-1}, and therefore its logarithm lies between -2 and -1. Therefore the characteristic is -2.

Hence, generally, if there are n zeros between the decimal point and the first significant figure, the number lies between $10^{-(n+1)}$ and 10^{-n}, and therefore its logarithm lies between $-(n+1)$ and $-n$. Therefore the characteristic is $-(n+1)$.

Thus we have the following rules for determining the characteristic of the logarithm of any number by inspection :

(*i*) *The characteristic of the logarithm of any number greater than unity is one less than the number of digits in the integral part of the number.*

(*ii*) *The characteristic of the logarithm of any number less than unity is negative, and numerically one greater than the number of zeros between the decimal point and the first significant figure of the number.*

Thus, the characteristic of $\log_{10}2708$ is 3, that of $\log_{10}753{\cdot}84$ is 2, and so on.

Again, the characteristic of $\log_{10}{\cdot}005$ is -3, for there are 2 zeros between the decimal point and the first significant figure 5.

Example 1. *Given* $\log_{10}2={\cdot}3010300$, *and* $\log_{10}3={\cdot}4771213$, *find the logarithm of* 3 *to the base* 2.

$$\log_2 3=\log_{10}3\div\log_{10}2 \qquad \text{[Art. 192]}$$

$$=\frac{{\cdot}4771213}{{\cdot}3010300}=\frac{4771213}{3010300}=1{\cdot}5849626.$$

Example 2. *Given log* $2 = \cdot 3010300$, *and log* $3 = \cdot 4771213$, *find the value of log* $\cdot 0072$.

$$\begin{aligned}\log \cdot 0072 &= \log \tfrac{72}{10000} = \log 72 - \log 10000 = \log (8 \times 9) - 4\\ &= \log (2^3 \times 3^2) - 4 = 3 \log 2 + 2 \log 3 - 4\\ &= 3 \times \cdot 3010300 + 2 \times \cdot 4771213 - 4 = 1\cdot 8573326 - 4\\ &= \bar{3}\cdot 8573326.\end{aligned}$$

Example 3. *Write down the characteristics of* (*i*) $log_7 1000$, (*ii*) $log_{10} 273\cdot 02$, *and* (*iii*) $log_{10} \cdot 0038$.

(*i*) Since $7^3 = 343$ and $7^4 = 2401$, and 1000 lies between 343 and 2401, $\log_7 1000$ lies between $\log_7 343$ and $\log_7 2401$, *i.e.*, between 3 and 4.

Hence, $\log_7 1000 = 3 +$ a decimal, and the required characteristic is 3.

(*ii*) Since the integral part 273 contains 3 digits, the required characteristic is 2.

(*iii*) Since there are 2 zeros between the decimal point and the first significant figure 3, the required characteristic is -3.

Example 4. *Find the number of digits in* 18^{30}, *given log* $2 = \cdot 30103$, *and log* $3 = \cdot 47712$.

Let $n = 18^{30}$.

$$\begin{aligned}\therefore \quad \log n &= 30 \log 18 = 30 \log (2 \times 3^2) = 30 \times \{\log 2 + 2 \log 3\}\\ &= 30 \times \{\cdot 30103 + 2 \times \cdot 47712\} = 30 \times 1\cdot 25527 = 37\cdot 65810\end{aligned}$$

Thus, the characteristic of $\log n$ is 37.

$\therefore$ the number of digits in n, *i.e.* 18^{30}, is 38.

Example 5. *Find the cube root of* $2\cdot 592$, *given log* $2 = \cdot 30103$, *log* $3 = \cdot 47712$ *and log* $13737 = 4\cdot 13788$.

Let $n = \sqrt[3]{(2\cdot 592)}$.

$$\begin{aligned}\text{Then } \log n &= \tfrac{1}{3} \log 2\cdot 592 = \tfrac{1}{3} \log \tfrac{2592}{1000} = \tfrac{1}{3} \log \frac{2^5 \times 3^4}{1000}\\ &= \tfrac{1}{3}\{5 \log 2 + 4 \log 3 - 3\} = \tfrac{1}{3}(\cdot 41363) = \cdot 13788\\ &= \log 1\cdot 3737 \qquad [\because \log 13737 = 4\cdot 13788]\\ \therefore \quad n &= 1\cdot 3737.\end{aligned}$$

Thus, the required cube root is $1\cdot 3737$.

Example 6. *Show without using tables that* $log_{10} 2$ *lies between* $\frac{1}{3}$ *and* $\frac{1}{4}$.

Let $\log_{10} 2 = x$, so that $10^x = 2$.

Then $10^{12x} = 2^{12} = 4096$.

$\therefore$ 10^{12x} lies between 1000 and 10000, *i.e.*, between 10^3 and 10^4.

$\therefore$ $12x$,, ,, 3 and 4

$\therefore$ x ,, ,, $\frac{1}{4}$ and $\frac{1}{3}$.

Thus what is required is proved.

EXERCISE 67.

1. Write down the characteristics of :
$\log_6 1003$, $\log_3 237$, $\log_{10} 385·6$, and $\log_{10} ·00034$.

2. Given $\log 2 = ·30103$, $\log 3 = ·47712$, and $\log 7 = ·84510$, find the logarithms of 784, 3·15, 52·5, ·0084 and ·0252.

3. Given $\log 2 = ·3010300$, find the logarithm of 1000 to the base 25.

4. Given $\log 2 = ·3010300$, find the number of digits in 2^{100}, and the number of zeros between the decimal point and the first significant digit in $(\frac{1}{2})^{100}$.

5. Show that 3^{100} lies between 2^{158} and 2^{159}, given $\log 2 = ·3010300$, and $\log 3 = ·4771213$.

6. Show without using the tables that $\log_{10} 3$ lies between $\frac{1}{3}$ and $\frac{1}{2}$.

7. Show without using the tables that $\log_{10} 2$ lies between $\frac{4}{16}$ and $\frac{5}{16}$.

8. Show without using the tables that $\log_{10} 7$ lies between $\frac{5}{6}$ and $\frac{6}{7}$.

9. Find the logarithm of $\sqrt{5}$ to the base ·008.

10. Given $\log 2 = ·30103$ and $\log 3 = ·47712$, find the number of digits in $2^{19} \times 3^{17}$.

11. Given $\log 72 = 1·85733$ and $\log 96 = 1·98227$, find $\log 2$ and $\log 3$.

12. Given $\log 245 = 2·38917$, and $\log 875 = 2·94201$, find $\log 5$ and $\log 7$.

13. Find what integral power of 2 is nearest to $(\frac{3}{2})^{12}$, given $\log 2 = ·30103$, and $\log 3 = ·47712$.

14. If $\frac{1}{2}(e^z + e^{-z}) = \frac{x}{a}$, show that $z = \log_e \frac{x \pm \sqrt{x^2 - a^2}}{a}$.

15. If $\frac{e^z - e^{-z}}{e^z + e^{-z}} = \frac{x}{a}$, show that $z = \frac{1}{2} \log_e \frac{a+x}{a-x}$.

16. If $\log m + \log n = \log (m+n)$, find m as a simple function of n. [Cal. Int. 1913]

197. Exponential Equations.

We solve below some Exponential Equations which involve logarithms in their solution.

Example 1. *Solve* $a^{x+3}=b^{2x+1}$.

Taking logarithms of both sides, we have

$$(x+3)\log a=(2x+1)\log b$$

or, $(\log a-2\log b)\,x=\log b-3\log a$

$$\therefore \quad x=\frac{\log b-3\log a}{\log a-2\log b}.$$

Example 2. *Solve* $2^x+27.2^{-x}=12$, *given* $\log 2$ *and* $\log 3$.

Multiplying by 2^x and transposing, we have

$$2^{2x}-12.2^x+27=0$$

or, $$(2^x-3)(2^x-9)=0.$$

$$\therefore \quad 2^x=3 \text{ or } 9.$$

(i) Let $2^x=3$; then $x\log 2=\log 3$.

$$\therefore \quad x=\frac{\log 3}{\log 2}=\frac{\cdot4771213}{\cdot3010300}=1\cdot58496\cdots$$

(ii) Let $2^x=9$; then $x\log 2=\log 9=2\log 3$.

$$\therefore \quad x=\frac{2\log 3}{\log 2}=\frac{2\times\cdot4771213}{\cdot3010300}=3\cdot16992\cdots$$

EXERCISE 68.

Given $\log 2=\cdot3010300$, $\log 3=\cdot4771213$, $\log 7=\cdot8450980$, solve the following equations :

1. $3^x=2$. [Cal. Int. 1927]
2. $5^x=3.2^{-(1+x)}$.
3. $5^{2x}=15.4^{3x}.6^x$.
4. $3^{x+1}=5^{x-1}$.
5. $4^{3x}=8.5^{x+2}$.
6. $3.5^{3-x}=3^{4-3x}.7^x$.
7. $a^{\sqrt{x}}=b$.
8. $a^{2x+3}=b^{3x+4}$.
9. $a^x+9a^{-x}=3(b^x+b^{-x})$.
10. $a^{b^x}=c$.
11. $8^x=9^y$; $3^{3x-1}=2^{2y-1}$.
12. $a^xb^y=1$, $a^yb^x=2$.
13. $2^x=3^y$, $2^{y+1}=3^{x-1}$. [Cal. Int. 1942]
14. $7^{3x+2}+4^{x+2}=7^{3x+1}+2^{2x+6}$. [Cal. Int. 1941]
15. $a^x=ab^y$, $b^x=ba^y$.
16. $a^xb^yc^z=b^xc^ya^z=c^xa^yb^z=m$.
17. If $\log(x^2y^3)=a$, and $\log\dfrac{x}{y}=b$, find $\log x$ and $\log y$. [Cal. Int. 1919]
18. If $a^{3-x}\,b^{5x}=a^{x+5}\,b^{3x}$, show that $x\log\left(\dfrac{b}{a}\right)=\log a$. [Cal. Int. 1937]

CHAPTER XVII

EXPONENTIAL THEOREM AND LOGARITHMIC SERIES

I. Exponential Theorem

198. Exponential Theorem.

If $n > 1$, we have, by the Binomial Theorem,

$$\left(1+\frac{1}{n}\right)^{nx}=1+nx.\frac{1}{n}+\frac{nx(nx-1)}{\underline{|2}}.\frac{1}{n^2}+\frac{nx(nx-1)(nx-2)}{\underline{|3}}.\frac{1}{n^3}+\ldots$$

$$=1+x+\frac{x\left(x-\frac{1}{n}\right)}{\underline{|2}}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\underline{|3}}+\ \ldots \qquad (1)$$

Putting $x=1$, we have

$$\left(1+\frac{1}{n}\right)^n=1+1+\frac{1\left(1-\frac{1}{n}\right)}{\underline{|2}}+\frac{1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\underline{|3}}+\ \ldots \qquad (2)$$

Now since $\left\{\left(1+\frac{1}{n}\right)^n\right\}^x=\left(1+\frac{1}{n}\right)^{nx}$, we have from (1) and (2)

$$\left\{1+1+\frac{1.\left(1-\frac{1}{n}\right)}{\underline{|2}}+\frac{1.\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\underline{|3}}+\ \ldots\ldots\right\}^x$$

$$=1+x+\frac{x\left(x-\frac{1}{n}\right)}{\underline{|2}}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\underline{|3}}+\ldots\ldots\ldots \qquad (3)$$

This result, being true for any value of n greater than 1, must be true when n is indefinitely increased. But when n is indefinitely increased, $\frac{1}{n}=0$, $\frac{2}{n}=0$, $\frac{3}{n}=0$, etc.

Hence, when n is infinite, from (3) we have

$$\left(1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\cdots\right)^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\cdots$$

The series $1+\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\ldots$ is usually denoted by the letter e, and hence we have

$$e^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\cdots \qquad (4)$$

Again, let $a=e^c$, so that $c=\log_e a$.

$$\therefore \quad a^x=e^{cx}=1+cx+\frac{c^2x^2}{\lfloor 2}+\frac{c^3x^3}{\lfloor 3}+\dots$$

$$=1+(\log_e a)x+(\log_e a)^2\frac{x^2}{\lfloor 2}+(\log_e a)^3\frac{x^3}{\lfloor 3}+\cdots \qquad (5)$$

This is called the *Exponential Theorem.*

NOTE. The above proof is not quite satisfactory, for two reasons viz. (i) it cannot be assumed that r/n tends to 0, when both r and n tend to ∞; and (ii) the sum of the limits ~~of an~~ infinite number of terms is not necessarily equal to the limit of their sum.

The theorem can, however, be established by a rigorous proof, which is beyond the scope of the present treatise.

199. Some Important Deductions.

We give below some important deductions from the results of the preceding article.

(i) Changing x into $-x$ in (4) and (5), we have

$$e^{-x}=1-x+\frac{x^2}{\lfloor 2}-\frac{x^3}{\lfloor 3}+\frac{x^4}{\lfloor 4}-\cdots\cdots$$

$$a^{-x}=1-(\log_e a)x+(\log_e a)^2\frac{x^2}{\lfloor 2}-(\log_e a)^3\frac{x^3}{\lfloor 3}+\cdots$$

Putting $x=1$, we have

$$e^{-1}=1-1+\frac{1}{\lfloor 2}-\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}-\frac{1}{\lfloor 5}+\cdots$$

$$=\frac{1}{\lfloor 2}-\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}-\frac{1}{\lfloor 5}+\cdots$$

$$a=1+\log_e a+\frac{(\log_e a)^2}{\lfloor 2}+\frac{(\log_e a)^3}{\lfloor 3}+\cdots$$

$$a^{-1}=1-\log_e a+\frac{(\log_e a)^2}{\lfloor 2}-\frac{(\log_e a)^3}{\lfloor 3}+\cdots$$

(*ii*) When n is indefinitely increased, the limit of $\left(1+\frac{1}{n}\right)^n=e$, and the limit of $\left(1+\frac{1}{n}\right)^{nx}=e^x$.

Putting $n=\frac{1}{p}$, we have the limit of $(1+p)^{\frac{1}{p}}=e$, when $p=0$.

Again, putting $nx=m$, so that $\frac{1}{n}=\frac{x}{m}$, and m becomes infinite at the same time as n, we have the limit of $\left(1+\frac{x}{m}\right)^m=e^x$, when m is infinite.

200. Convergency of *e*.

Since $e = 1 + \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \ldots = 2 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \ldots$

it is obvious that e is greater than 2.

Again, since $\frac{1}{\underline{|3}} = \frac{1}{2} \cdot \frac{1}{3}$; *i.e.*, $< \frac{1}{2} \cdot \frac{1}{2}$ or $\frac{1}{2^2}$;

$\frac{1}{\underline{|4}} = \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4}$; *i.e.*, $< \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$ or $\frac{1}{2^3}$.

and so on, we have

$$1 + \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \cdots < 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

i.e., $\quad < 1 + \frac{1}{1 - \frac{1}{2}}$ or $1 + \frac{1}{\frac{1}{2}}$ or $1 + 2$ or 3.

$\therefore$ e lies between 2 and 3.

This shows that the series for e is convergent.

Alternative Proof:

Since $u_{n+1} : u_n = \frac{1}{\underline{|n}} \div \frac{1}{\underline{|n-1}} = \frac{1}{n}$; it is evident that the test-ratio <1, and hence the series is convergent. [Art. 146]

201. Incommensurability of *e*.

If possible, let e be equal to a commensurable fraction $\frac{m}{n}$, where m and n are positive integers. Then

$$\frac{m}{n} = \left\{1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \cdots + \frac{1}{\underline{|n}}\right\} + \frac{1}{\underline{|n+1}} + \frac{1}{\underline{|n+2}} + \ldots$$

Multiply both sides by $\underline{|n}$; then

$$m\,\underline{|n-1} = \text{an integer} + \frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \ldots\ldots$$

Now $\frac{1}{n+1} + \frac{1}{(n+1)(n+2)} + \frac{1}{(n+1)(n+2)(n+3)} + \cdots\cdots$

$$< \frac{1}{n+1} + \frac{1}{(n+1)^2} + \frac{1}{(n+1)^3} + \ldots\ldots$$

i.e., $\quad < \frac{1}{n+1} \div \left(1 - \frac{1}{n+1}\right)$

i.e., $\quad < \frac{1}{n}$ which is a proper fraction.

$\therefore$ the integer $m\,\underline{|n-1}$ = an integer + a proper fraction, which is impossible.

$\therefore$ e cannot be of the form $\frac{m}{n}$, *i.e.*, it is incommensurable.

202. Numerical Value of e.

Since $e = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \frac{1}{\lfloor 4} + \cdots$, we have

$$\begin{aligned}
1 &= 1 \\
\frac{1}{\lfloor 1} &= 1 \\
\frac{1}{\lfloor 2} &= \cdot 5 \\
\frac{1}{\lfloor 3} &= \frac{1}{\lfloor 2} \div 3 = \cdot 16666667 \\
\frac{1}{\lfloor 4} &= \frac{1}{\lfloor 3} \div 4 = \cdot 04166667 \\
\frac{1}{\lfloor 5} &= \frac{1}{\lfloor 4} \div 5 = \cdot 00833333 \\
\frac{1}{\lfloor 6} &= \frac{1}{\lfloor 5} \div 6 = \cdot 00138889 \\
\frac{1}{\lfloor 7} &= \frac{1}{\lfloor 6} \div 7 = \cdot 00019841 \\
\frac{1}{\lfloor 8} &= \frac{1}{\lfloor 7} \div 8 = \cdot 00002480 \\
\frac{1}{\lfloor 9} &= \frac{1}{\lfloor 8} \div 9 = \cdot 00000276 \\
\frac{1}{\lfloor 10} &= \frac{1}{\lfloor 9} \div 10 = \cdot 00000028 \\
\frac{1}{\lfloor 11} &= \frac{1}{\lfloor 10} \div 11 = \cdot 00000003 \\
& \qquad\qquad \overline{2 \cdot 71828184}
\end{aligned}$$

Hence the value of e correct to 7 places of decimals is 2·7182818.

203. Convergency of the Exponential Series.

(i) Since $e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots$ we have

$$u_{n+1} : u_n = \frac{x^n}{\lfloor n} \div \frac{x^{n-1}}{\lfloor n-1} = \frac{x}{n},$$

which is less than 1 for any finite value of x, when n is indefinitely increased. Hence the expansion of e^x is convergent, for all finite values of x. [Art. 146.]

(ii) Since $a^x = 1 + (\log_e a)x + (\log_e a)^2 \frac{x^2}{\underline{|2}} + (\log_e a)^3 \frac{x^3}{\underline{|3}} + \cdots$

we have $u_{n+1} : u_n = (\log_e a)^n \frac{x^n}{\underline{|n}} \div (\log_e a)^{n-1} \frac{x^{n-1}}{\underline{|n-1}} = (\log_e a) \cdot \frac{x}{n}$

which is less than 1 for any finite value of x, when n is indefinitely increased. Therefore the expansion of a^x is convergent for all finite values of x. [Art. 146.]

204. Factorials of Negative Integers.

Since $\underline{|n} = n\,\underline{|n-1}$, we have $\underline{|n-1} = \frac{\underline{|n}}{n}$.

Putting $n=1$, we get $\underline{|0} = \underline{|1} \div 1 = 1$;

" $n=0$, " $\underline{|-1} = \underline{|0} \div 0 = 1 \div 0 = \infty$;

" $n=-1$, " $\underline{|-2} = \underline{|-1} \div (-1) = \infty \div (-1) = -\infty$;

" $n=-2$, " $\underline{|-3} = \underline{|-2} \div (-2) = (-\infty) \div (-2) = \infty$;

" $n=-3$, " $\underline{|-4} = \underline{|-3} \div (-3) = \infty \div (-3) = -\infty$;

and so on.

Thus *the factorial of a negative integer is infinite, and is positive or negative according as the integer is odd or even.*

Example 1. *Show that* $e^{-1} = \frac{2}{\underline{|3}} + \frac{4}{\underline{|5}} + \frac{6}{\underline{|7}} + \ldots$ *to infinity.*

[*Cal. Int.* 1937]

$$e^{-1} = 1 - 1 + \frac{1}{\underline{|2}} - \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} - \frac{1}{\underline{|5}} + \frac{1}{\underline{|6}} - \frac{1}{\underline{|7}} + \cdots$$

$$= \frac{3-1}{\underline{|3}} + \frac{5-1}{\underline{|5}} + \frac{7-1}{\underline{|7}} + \cdots$$

$$= \frac{2}{\underline{|3}} + \frac{4}{\underline{|5}} + \frac{6}{\underline{|7}} + \cdots$$

Example 2. *Show that*

$$\frac{1}{\underline{|2}} + \frac{1+2}{\underline{|3}} + \frac{1+2+3}{\underline{|4}} + \frac{1+2+3+4}{\underline{|5}} + \cdots = \frac{e}{2}.$$

[*Cal. Int.* 1942, 1945]

The nth term of the given series

$$= \frac{1+2+3+\ldots+n}{\underline{|n+1}} = \frac{\frac{1}{2}n(n+1)}{\underline{|n+1}} = \frac{1}{2} \cdot \frac{1}{\underline{|n-1}}.$$

Making $n = 1, 2, 3, 4, \ldots\ldots$ and observing that $\underline{|0} = 1$, the given series becomes

$$\frac{1}{2}\left(1 + \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \ldots\ldots\right).$$

Example 3. *Find the sum of the infinite series*

$$1+\frac{2^2}{\underline{|2}}+\frac{3^2}{\underline{|3}}+\frac{4^2}{\underline{|4}}+\ldots\ldots$$

The nth term of the given series

$$=\frac{n^2}{\underline{|n}}=\frac{n}{\underline{|n-1}}=\frac{(n-1)+1}{\underline{|n-1}}=\frac{1}{\underline{|n-2}}+\frac{1}{\underline{|n-1}}.$$

Making $n=1, 2, 3, 4, \cdots$ and observing, that, $\underline{|-1}=\infty$, and $\underline{|0}=1$ so that $\dfrac{1}{\underline{|-1}}=0$ and $\dfrac{1}{\underline{|0}}=1$, we have the given series

$$=1+\left(\frac{1}{\underline{|0}}+\frac{1}{\underline{|1}}\right)+\left(\frac{1}{\underline{|1}}+\frac{1}{\underline{|2}}\right)+\left(\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}\right)+\cdots$$

$$=2+\frac{2}{\underline{|1}}+\frac{2}{\underline{|2}}+\frac{2}{\underline{|3}}+\cdots$$

$$=2e.$$

Example 4. *Find the co-efficient of x^r in the expansion of* $(a+bx+cx^2)e^x$.

$$(a+bx+cx^2)e^x=(a+bx+cx^2)\left(1+x+\frac{x^2}{\underline{|2}}+\cdot\right.$$
$$\left.+\frac{x^{r-2}}{\underline{|r-2}}+\frac{x^{r-1}}{\underline{|r-1}}+\frac{x^r}{\underline{|r}}+\cdots\cdots\right)$$

$\therefore$ the required co-eff. of $x^r=\dfrac{a}{\underline{|r}}+\dfrac{b}{\underline{|r-1}}+\dfrac{c}{\underline{|r-2}}$

$$=\frac{1}{\underline{|r}}\{a+br+cr(r-1)\}.$$

Example 5. *Find the value of $e^{\frac{1}{10}}$ correct to 5 places of decimals.*

Since $e^x=1+x+\dfrac{x^2}{\underline{|2}}+\dfrac{x^3}{\underline{|3}}+\cdots\cdots$, we have, putting $x=\frac{1}{10}$,

$$e^{\frac{1}{10}}=1+\frac{1}{10}+\frac{1}{10^2.\underline{|2}}+\frac{1}{10^3.\underline{|3}}+\frac{1}{10^4.\underline{|4}}+\cdots$$

$$=1+{\cdot}1+{\cdot}005+{\cdot}00016666\ldots+{\cdot}00000416\cdots\cdots+\cdots\cdots$$

$$=1{\cdot}10517082\cdots=1{\cdot}10517 \text{ (up to 5 places).}$$

EXERCISE 69.

Express in terms of e :

1. $1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 4}+\frac{1}{\lfloor 6}+\cdots$ **2.** $1+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 5}+\frac{1}{\lfloor 7}+\ldots$

3. $\left(1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\cdots\right)\left(1-\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}-\cdots\right)$ [Cal. Int. 1938]

4. $\frac{1.2}{\lfloor 1}+\frac{2.3}{\lfloor 2}+\frac{3.4}{\lfloor 3}+\frac{4.5}{\lfloor 4}+\cdots$

5. $\frac{1^3}{\lfloor 1}+\frac{2^3}{\lfloor 2}+\frac{3^3}{\lfloor 3}+\frac{4^3}{\lfloor 4}+\ldots$ [Cal. Int. 1939] **6.** $1+\frac{3}{\lfloor 1}+\frac{5}{\lfloor 2}+\frac{7}{\lfloor 3}+\frac{9}{\lfloor 4}+\cdots$

7. $\frac{5}{\lfloor 1}+\frac{9}{\lfloor 2}+\frac{14}{\lfloor 3}+\frac{20}{\lfloor 4}+\cdots$ **8.** $\frac{3}{\lfloor 1}+\frac{10}{\lfloor 2}+\frac{21}{\lfloor 3}+\frac{36}{\lfloor 4}+\ldots$

9. $(a^2-b^2)+\frac{1}{\lfloor 2}(a^4-b^4)+\frac{1}{\lfloor 3}(a^6-b^6)+\ldots$

10. $1+\frac{1+a}{\lfloor 2}+\frac{1+a+a^2}{\lfloor 3}+\frac{1+a+a^2+a^3}{\lfloor 4}+\cdots$ [Cal. F. A. 1888]

11. Show that

$$\frac{e-1}{e+1}=\left\{\frac{1}{\lfloor 2}+\frac{1}{\lfloor 4}+\frac{1}{\lfloor 6}+\cdots\right\}\div\left\{\frac{1}{\lfloor 1}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 5}+\frac{1}{\lfloor 7}+\ldots\right\}.$$

[Cal. Int. 1934]

12. Show that

$$\frac{e^2+1}{e^2-1}=\left\{1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 4}+\frac{1}{\lfloor 6}+\ldots\right\}\div\left\{\frac{1}{\lfloor 1}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 5}+\frac{1}{\lfloor 7}+\cdots\right\}.$$

13. Show that $1+e^{-1}=\frac{2\frac{1}{2}}{\lfloor 1}-\frac{3\frac{1}{3}}{\lfloor 2}+\frac{4\frac{1}{4}}{\lfloor 3}-\frac{5\frac{1}{5}}{\lfloor 4}+\cdots$

14. Show that $\frac{1}{e}=\frac{1}{1.3}+\frac{1}{1.2.3.5}+\frac{1}{1.2.3.4.5.7}+\cdots$

15. Show that $1+\frac{1+2}{\lfloor 2}+\frac{1+2+3}{\lfloor 3}+\cdots=\frac{3e}{2}$ [Cal. Int. 1935]

16. Show that $\frac{2}{\lfloor 1}+\frac{4}{\lfloor 3}+\frac{6}{\lfloor 5}+\frac{8}{\lfloor 7}+\cdots$ to inf. $=e$. [Cal. Int. 1936]

17. Show that $e^2-e=1+\frac{1+2}{\lfloor 2}+\frac{1+2+2^2}{\lfloor 3}+\cdots\cdots$

[Cal. Int. 1929]

18. Show that $n+\frac{1}{n}=2\left\{1+\frac{(\log_e n)^2}{\underline{|2}}+\frac{(\log_e n)^4}{\underline{|4}}+\cdots\right\}$.

19. Show that

$$\left(1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|4}}+\frac{1}{\underline{|6}}+\cdots\right)^2-\left(1+\frac{1}{\underline{|3}}+\frac{1}{\underline{|5}}+\frac{1}{\underline{|7}}+\cdots\right)^2=1.$$

20. Show that

(i) $\left\{1+\frac{x^2}{\underline{|2}}+\frac{x^4}{\underline{|4}}+\frac{x^6}{\underline{|6}}+\right\}^2-\left\{x+\frac{x^3}{\underline{|3}}+\frac{x^5}{\underline{|5}}+\frac{x^7}{\underline{|7}}+\cdots\right\}^2=1$.

(ii) $\left\{1-\frac{x^2}{\underline{|2}}+\frac{x^4}{\underline{|4}}-\frac{x^6}{\underline{|6}}+\cdots\right\}^2+\left\{x-\frac{x^3}{\underline{|3}}+\frac{x^5}{\underline{|5}}-\frac{x^7}{\underline{|7}}+\cdots\right\}^2=1$.

21. Find the value of

(i) $\frac{1}{\sqrt{e}}$ correct to two places of decimals. [Cal. F. A. 1889]

(ii) $\frac{1}{\sqrt[5]{e}}$ correct to four places of decimals. [Cal. Int. 1936]

22. Expand e^{e^x} in ascending powers of x up to x^4. [Cal. Int. 1938]

23. Find the co-efficient of x^r in the expansion of

$$\frac{a+bx+cx^2}{e^x}.$$

24. Show that the co-efficient of x^n in the infinite series $\frac{a+bx}{\underline{|1}}+\frac{(a+bx)^2}{\underline{|2}}+\ldots+\frac{(a+bx)^n}{\underline{|n}}+\cdots$ is $\frac{b^n}{\underline{|n}}e^a$.

25. Show that the limit of $\sqrt[x]{\frac{1+x}{1-x}}$ when $x=0$, is e^2.

26. Show that

$$x^{\log_e x}=1+(\log_e)x^2+\frac{(\log_e x)^4}{1.2}+\frac{(\log_e x)^6}{1.2.3}+\cdots\text{to infinity.}$$

27. Find the limit of $\left(1+\frac{n}{a}\right)^{\frac{x}{n}}$, when $n=0$.

28. Show that $\left(1+\frac{1}{a}\right)^a>\left(1+\frac{1}{b}\right)^b$, when a and b are positive integers, and $a>b$; and hence show that

$$(1\tfrac{1}{2})^2<(1\tfrac{1}{3})^3<(1\tfrac{1}{4})^4<\ldots\ldots<e.$$

II. Logarithmic Series. 326

205. Logarithmic Series.

By the Exponential Theorem, for all positive values of a, we have

$$a^y = 1 + y \log_e a + \frac{y^2(\log_e a)^2}{\lfloor 2} + \frac{y^3(\log_e a)^3}{\lfloor 3} + \ldots\ldots$$

Putting $a = 1 + x$, this becomes

$$(1+x)^y = 1 + y \log_e(1+x) + \frac{y^2}{\lfloor 2}[\log_e(1+x)]^2 + \cdots \qquad (1)$$

Also, by the Binomial theorem, if $x < 1$ numerically, for all values of y, we have

$$(1+x)^y = 1 + yx + \frac{y(y-1)}{\lfloor 2}x^2 + \frac{y(y-1)(y-2)}{\lfloor 3}x^3 + \cdots \qquad (2)$$

Hence, from (1) and (2), if $x < 1$, we have

$$1 + y \log_e (1+x) + \frac{y^2}{\lfloor 2}[\log_e (1+x)]^2 + \ldots$$

$$= 1 + yx + \frac{y(y-1)}{\lfloor 2}x^2 + \frac{y(y-1)(y-2)}{\lfloor 3}x^3 + \cdots$$

This being an identity we can equate the co-efficient of y on either side. Hence

$$\log_e (1+x) = x - \frac{x^2}{\lfloor 2} + \frac{1.2}{\lfloor 3}x^3 - \frac{1.2.3}{\lfloor 4}x^4 + \ldots$$

i.e., $$\log_e (1+x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \tfrac{1}{4}x^4 + \ldots \qquad (3)$$

when $x < 1$ numerically. This series is called the *Logarithmic Series.*

Again, since the expansion of $\log_e(1+x)$ holds when x is numerically less than 1, we have, putting $-x$ for x in (3),

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots\ldots \qquad (4)$$

Subtracting (4) from (3),

$$\log_e(1+x) - \log_e (1-x) = \left\{x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\right\}$$

$$-\left\{-x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots\right\}$$

or, $$\log_e \frac{1+x}{1-x} = 2\left\{x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\ldots\right\} \qquad (5)$$

206. Convergency of the Logarithmic Series.

The Logarithmic series is of course convergent when $x<1$; and it holds good even when $x=1$; for putting $x=1$, the series becomes $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$, and it may be written in each of the following forms :

$$\left(1-\tfrac{1}{2}\right)+\left(\tfrac{1}{3}-\tfrac{1}{4}\right)+\left(\tfrac{1}{5}-\tfrac{1}{6}\right)+\cdots\cdots=\frac{1}{1.2}+\frac{1}{3.4}+\frac{1}{5.6}+\cdots\cdots \qquad (1)$$

$$1-\left(\tfrac{1}{2}-\tfrac{1}{3}\right)-\left(\tfrac{1}{4}-\tfrac{1}{5}\right)-\left(\tfrac{1}{6}-\tfrac{1}{7}\right)\cdots\cdots=1-\frac{1}{2.3}-\frac{1}{4.5}-\frac{1}{6.7}-\cdots\cdots \qquad (2)$$

From (1), we see that the sum of any number of terms of the series is a poisitve quantity, and from (2) that the sum of any number of terms is less than 1. Hence the sum of the series is a finite quantity and therefore the series is convergent.

$$\log_e 2=1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\cdots\cdots$$

The series, however, is not convergent when $x>1$. Further, the series for $\log_e(1-x)$ is not convergent when $x=1$ or $x>1$.

207. Derived series for Logarithmic calculation.

We deduce below some rapidly converging series which are very convenient for the calculation of logarithms of any number.

Thus in Art. 205 (5), if $\dfrac{1+x}{1-x}=\dfrac{m}{n}$ so that $x=\dfrac{m-n}{m+n}$, we have

$$\log_e\frac{m}{n}=2\left\{\frac{m-n}{m+n}+\frac{1}{3}\left(\frac{m-n}{m+n}\right)^3+\frac{1}{5}\left(\frac{m-n}{m+n}\right)^5+\cdots\cdots\cdots\right\} \qquad (1)$$

This becomes, if $n=1$,

$$\log_e m=2\left\{\frac{m-1}{m+1}+\frac{1}{3}\left(\frac{m-1}{m+1}\right)^3+\frac{1}{5}\left(\frac{m-1}{m+1}\right)^5+\cdots\cdots\cdots\right\} \qquad (2)$$

These series may be used to calculate the logarithms of small numbers like 2, 3, etc.

Again, if we put $m=n+1$ in (1), so that $\dfrac{m-n}{m+n}=\dfrac{1}{2n+1}$;

$$\log_e\frac{n+1}{n}=2\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\cdots\right\}$$

$$\text{or, } \log_e(n+1)=\log_e n+2\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\cdots\right\} \qquad (3)$$

This is a rapidly converging series and may be conveniently used to deduce the logarithm of any number from that of the preceding number.

208. The Two Systems of Logarithms : Common and Napierian.

It follows from what has already been proved in Art. 192, that if n be any number,

$$\log_{10} n = \log_e n \times \frac{1}{\log_e 10};$$

and by means of this relation, the logarithm of any number to the base 10 (*i.e.*, the Common system) is obtained from the logarithm of the same number to the base e (*i.e.*, the Napierian system) by multiplying it by the constant multiplier $\frac{1}{\log_e 10}$. This constant multiplier $\frac{1}{\log_e 10}$, which is called the *modulus* of the Common system of logarithms

$$= \frac{1}{2{\cdot}30258509} = {\cdot}43429448.$$

Thus, if we denote the *modulus* by μ (a Greek letter called *mu*) we have from Art. 207 (3),

$$\log_{10}(n+1) - \log_{10} n = \mu \log_e(n+1) - \mu \log_e n$$

$$= 2\mu\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \cdots\right\}$$

where $\mu = {\cdot}43429448$.

This formula enables us to calculate logarithms of numbers to the base 10. The method is illustrated below.

Example 1. *Calculate the value of* $\log_{10} 2$.

putting $n=1$ in Art. 207 (3), since $\log_e 1 = 0$, we have

$$\log_e 2 = 2\left\{\frac{1}{3} + \frac{1}{3.3^3} + \frac{1}{5.3^5} + \frac{1}{7.3^7} + \cdots\right\}$$

$$= 2[{\cdot}333333333\cdots + \tfrac{1}{3} \times {\cdot}037037037\cdots + \tfrac{1}{5} \times {\cdot}004115226\cdots$$
$$+ \tfrac{1}{7} \times {\cdot}000457247\cdots + \tfrac{1}{9} \times {\cdot}000050805\cdots + \tfrac{1}{11} \times {\cdot}000005645\cdots$$
$$+ \tfrac{1}{13} \times {\cdot}000000627 + \tfrac{1}{15} \times 000000070\cdots + \tfrac{1}{17} \times {\cdot}000000008\cdots]$$

$= {\cdot}693147178 = {\cdot}69314718$ (correct to 8 places of decimals)

$$\therefore \log_{10} 2 = \log_e 2 \times \frac{1}{\log_e 10} = {\cdot}69314718 \times {\cdot}43429448$$

$= {\cdot}3010300$, correct to 7 places of decimals.

209. We work out below some miscellaneous examples relating to the Exponential and Logarithmic series.

Example 1. *Show that*

$$\log_e(n+1)-\log_e n=\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\frac{1}{4n^4}+\cdots$$

$$\log_e(n+1)-\log_e n=\log_e\frac{n+1}{n}=\log_e\left[1+\frac{1}{n}\right]$$

$$=\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\frac{1}{4n^4}+\cdots$$

Example 2. *Show that* $\log_e 2=\frac{1}{2}+\frac{1}{1.2.3}+\frac{1}{3.4.5}+\frac{1}{5.6.7}+\cdots$

Putting $x=1$ in the series

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$$

we have

$$\log_e 2=1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\cdots$$
$$=(1-\tfrac{1}{2})+(\tfrac{1}{3}-\tfrac{1}{4})+(\tfrac{1}{5}-\tfrac{1}{6})+\cdots$$
$$=\frac{1}{1\cdot2}+\frac{1}{3\cdot4}+\frac{1}{5\cdot6}+\cdots \qquad (1)$$

Also

$$\log_e 2=1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}-\tfrac{1}{6}+\tfrac{1}{7}-\cdots$$
$$=1-(\tfrac{1}{2}-\tfrac{1}{3})-(\tfrac{1}{4}-\tfrac{1}{5})-(\tfrac{1}{6}-\tfrac{1}{7})-\cdots$$
$$=1-\frac{1}{2\cdot3}-\frac{1}{4\cdot5}-\frac{1}{6\cdot7}-\cdots \qquad (2)$$

Adding (1) and (2),

$$2\log_e 2=1+\left(\frac{1}{1\cdot2}-\frac{1}{2\cdot3}\right)+\left(\frac{1}{3\cdot4}-\frac{1}{4\cdot5}\right)+\left(\frac{1}{5\cdot6}-\frac{1}{6\cdot7}\right)+\cdots$$
$$=1+\frac{2}{1\cdot2\cdot3}+\frac{2}{3\cdot4\cdot5}+\frac{2}{5\cdot6\cdot7}+\cdots$$
$$\therefore\quad \log_e 2=\tfrac{1}{2}+\frac{1}{1\cdot2\cdot3}+\frac{1}{3\cdot4\cdot5}+\frac{1}{5\cdot6\cdot7}+\cdots$$

Example 3. *Show that*

$$\log_e\sqrt{2}=\frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right)-\frac{1}{4}\left(\frac{1}{2^2}+\frac{1}{3^2}\right)+\frac{1}{6}\left(\frac{1}{2^3}+\frac{1}{3^3}\right)-\cdots$$

$$\log_e\sqrt{2}=\tfrac{1}{2}\log_e 2=\tfrac{1}{2}\log_e(\tfrac{3}{2}\times\tfrac{4}{3})=\tfrac{1}{2}\{\log_e(1+\tfrac{1}{2})+\log_e(1+\tfrac{1}{3})\}$$

$$=\frac{1}{2}\left\{\frac{1}{2}-\frac{1}{2.2^2}+\frac{1}{3.2^3}-\cdots\right\}+\frac{1}{2}\left\{\frac{1}{3}-\frac{1}{2.3^2}+\frac{1}{3.3^3}-\cdots\right\}$$

$$=\frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right)-\frac{1}{4}\left(\frac{1}{2^2}+\frac{1}{3^2}\right)+\frac{1}{6}\left(\frac{1}{2^3}+\frac{1}{3^3}\right)-\cdots$$

Example 4. *Expand* $\log_e \dfrac{1+x+x^2}{1-x+x^2}$ *in ascending powers of* x.

$$\log_e \frac{1+x+x^2}{1-x+x^2} = \log_e \left\{\frac{1-x^3}{1+x^3} \cdot \frac{1+x}{1-x}\right\} = \log_e \frac{1-x^3}{1+x^3} + \log_e \frac{1+x}{1-x}$$
$$= 2[-x^3 - \tfrac{1}{3}x^9 - \tfrac{1}{5}x^{15} - \cdots] + 2[x + \tfrac{1}{3}x^3 + \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 + \tfrac{1}{9}x^9 + \cdots]$$
$$= 2(x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5 + \tfrac{1}{7}x^7 - \tfrac{2}{9}x^9 + \cdots)$$

Example 5. *Show that if* x *be positive,*

$$\log_e x = \frac{x-1}{x+1} + \frac{1}{2} \cdot \frac{x^2-1}{(x+1)^2} + \frac{1}{3} \cdot \frac{x^3-1}{(x+1)^3} + \cdots\cdots$$

We have $x = \dfrac{x}{x+1} \cdot \dfrac{x+1}{1} = \dfrac{x}{x+1} \div \dfrac{1}{x+1} = \left(1 - \dfrac{1}{x+1}\right) \div \left(1 - \dfrac{x}{x+1}\right)$

$$\therefore \log_e x = \log_e\left(1 - \frac{1}{x+1}\right) - \log_e\left(1 - \frac{x}{x+1}\right)$$
$$= \left\{-\frac{1}{x+1} - \frac{1}{2(x+1)^2} - \frac{1}{3(x+1)^3} - \cdots\right\}$$
$$+ \left\{\frac{x}{x+1} + \frac{x^2}{2(x+1)^2} + \frac{x^3}{3(x+1)^3} + \cdots\right\}$$
$$= \frac{x-1}{x+1} + \frac{1}{2} \cdot \frac{x^2-1}{(x+1)^2} + \frac{1}{3} \cdot \frac{x^3-1}{(x+1)^3} + \cdots$$

Example 6. *If* α, β *be the roots of the equation* $ax^2+bx+c=0$, *show that* $\log_e (a+bx+cx^2)$

$$= \log_e a - \left\{(\alpha+\beta)x + \frac{\alpha^2+\beta^2}{2}x^2 + \frac{\alpha^3+\beta^3}{3}x^3 + \cdots\right\}$$

$$a+bx+cx^2 = a\left\{1 + \frac{b}{a}x + \frac{c}{a}x^2\right\}$$
$$= a\{1-(\alpha+\beta)x + \alpha\beta x^2\} \quad \left[\because \alpha+\beta = -\frac{b}{a},\ \alpha\beta = \frac{c}{a}\right]$$
$$= a(1-\alpha x)(1-\beta x)$$

$$\therefore \log_e(a+bx+cx^2) = \log_e a + \log_e(1-\alpha x) + \log_e(1-\beta x)$$
$$= \log_e a - \alpha x - \frac{\alpha^2x^2}{2} - \frac{\alpha^3x^3}{3} - \cdots - \beta x - \frac{\beta^2x^2}{2} - \frac{\beta^3x^3}{3} - \cdots$$
$$= \log_e a - \left\{(\alpha+\beta)x + \frac{\alpha^2+\beta^2}{2}x^2 + \frac{\alpha^3+\beta^3}{3}x^3 + \cdots\right\}.$$

EXERCISE 70.

1. Show that $\log_e(n+1)-\log_e(n-1)=2\left(\frac{1}{n}+\frac{1}{3n^3}+\frac{1}{5n^5}+\cdots\right)$

2. Prove that $\frac{1}{n+1}+\frac{1}{2(n+1)^2}+\frac{1}{3(n+1)^3}+\cdots$

$$=\frac{1}{n}-\frac{1}{2n^2}+\frac{1}{3n^3}-\cdots$$ [Cal. Int. 1914, 1945]

3. Prove that

$$\log_e a-\log_e x=\frac{a-x}{a}+\frac{1}{2}\left(\frac{a-x}{a}\right)^2+\frac{1}{3}\left(\frac{a-x}{a}\right)^3+\cdots$$

4. Prove that $\log_e\left(\frac{1}{1-x}\right)=x+\frac{x^2}{2}+\frac{x^3}{3}+\cdots$

5. Sum to infinity the following series :

(i) $\frac{2}{1}x+\frac{3}{2}x^2+\frac{4}{3}x^3+\frac{5}{4}x^4+\cdots$

(ii) $\frac{1}{2}x+\frac{2}{3}x^2+\frac{3}{4}x^3+\frac{4}{5}x^4+\ldots$

6. Expand $\log_e(1-5x+6x^2)$ in ascending powers of x, and find the co-efficient of x^n.

7. Expand $\log_e(1+x+x^2)$ in ascending powers of x, and write down the co-efficient of x^n.

8. Expand $\log_e(1+x+x^2+x^3)$ in powers of x, and find the co-efficient of x^{2n} and x^{2n+1}. [Cal. F. A. 1891]

9. If α, β be the roots of $x^2+px+q=0$, show that

$$\log(1-px+qx^2)=(\alpha+\beta)x-\tfrac{1}{2}(\alpha^2+\beta^2)x^2+\tfrac{1}{3}(\alpha^3+\beta^3)x^3-\cdots$$

10. If $y=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$, show that

$$x=y+\frac{y^2}{\lfloor 2}+\frac{y^3}{\lfloor 3}+\frac{y^4}{\lfloor 4}+\ldots$$ [Cal. Int. 1931]

11. Show that

$$\log_e\left(1+\frac{1}{x}\right)=1-\frac{1}{1.2(1+x)}-\frac{1}{2.3(1+x)^2}-\frac{1}{3.4(1+x)^3}-\cdots$$

12. If x, y, z are in Harmonical Progression and in descending order of magnitude, show that

$$\log_e y-\log_e z=\left(\frac{z}{y}-\frac{y}{x}\right)+\frac{1}{2}\left(\frac{z^2}{y^2}-\frac{y^2}{x^2}\right)+\frac{1}{3}\left(\frac{z^3}{y^3}-\frac{y^3}{x^3}\right)+\cdots$$

13. Prove that $\log_e 3 = 1 + \frac{1}{3.2^2} + \frac{1}{5.2^4} + \frac{1}{7.2^6} + \cdots$

[Cal. F. A. 1903]

14. Show that $\log_e 3 - \log_e 2 = \frac{1}{2} - \frac{1}{2.2^2} + \frac{1}{3.2^3} - \frac{1}{4.2^4} + \cdots$

[Cal. F. A. 1903]

15. Sum the following series to infinity :

(i) $\frac{1}{2.3} + \frac{1}{4.5} + \frac{1}{6.7} + \cdots\cdots$ [Cal. Int. 1940].

(ii) $\frac{1}{2.3.4} + \frac{1}{4.5.6} + \frac{1}{6.7.8} + \cdots$

(iii) $\frac{1}{1.2.3.4} + \frac{1}{3.4.5.6} + \frac{1}{5.6.7.8} + \cdots$

(iv) $\frac{1}{1.2.3.4} + \frac{4}{3.4.5.6} + \frac{9}{5.6.7.8} + \cdots$

16. Prove that

$$\log_e\{(1+x)^{1+x}(1-x)^{1-x}\} = 2\left\{\frac{x^2}{1.2} + \frac{x^4}{3.4} + \frac{x^6}{5.6} + \cdots\cdots\right\}$$

17. If x is numerically less than 1, show that

$$\frac{x}{1+x^2} + \frac{1}{3}\left(\frac{x}{1+x^2}\right)^3 + \frac{1}{5}\left(\frac{x}{1+x^2}\right)^5 + \cdots = x - \frac{2x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{2x}{9}$$

18. Prove that $\log(x+n) = \log x + \log\left(1 + \frac{1}{x}\right) + \log\left(1 + \frac{1}{1+x}\right)$

$$+ \log\left(1 + \frac{1}{2+x}\right) + \cdots + \log\left(1 + \frac{1}{n-1+x}\right).$$

19. Expand $\{\log_e(1+x)\}^2$ in ascending powers of x, when $x<1$.

20. Show that, if x be positive and <2,

$$(1-x) + (1+\tfrac{1}{2})(1-x)^2 + (\tfrac{1}{2}+\tfrac{1}{3})(1-x)^3 + (\tfrac{1}{3}+\tfrac{1}{4})(1-x)^4 + \cdots = (x-2)\log_e x.$$

21. Point out the fallacy in the following reasoning :

If $S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$, then

$$\log_e 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots = S - 2.\tfrac{1}{2}\,S = S - S = 0.$$

CHAPTER XVIII

INTEREST AND ANNUITIES.

210. Definitions.

Interest is the consideration paid for the use of money, and the *rate of interest* is the sum paid for the use of a certain sum for a certain time, as of £1 or Re. 1 for 1 year or 1 month. Usually however, the rate is calculated for £100 or Rs. 100 for one year, and is called the *rate per cent. per annum.*

The *Principal* is the sum of money for the use of which interest is paid ; and the *Amount* at any time is the sum of the Principal and the Interest accruing up to that time.

When the Principal alone produces interest it is called *Simple Interest.* When, however, the interest is added to the Principal as soon as it falls due, and the whole then produces interest, it is called *Compound Interest.*

Discount is the reduction made from a sum of money when it is paid before it becomes due ; and if the discount be subtracted from the sum, the remainder due is called the *Present Worth* or *Present Value* of the sum.

Annuity means any sum of money, in the shape of Interest or Rent or Pension, payable *annually.* The actual payments may be made at any other definite intervals, say, *half-yearly*, *quarterly*, etc.

211. Simple Interest and its Formulae.

If P denotes the principal of sum lent, r the interest on Re. 1 per year, n the number of years the interest accrues, I the total interest accrued on the sum lent, and A the amount at the end of n years, we have

Pr = the interest on P per year

and Pnr = „ „ for n years.

Hence, $I = Pnr$; and $A = P + I = P(1 + nr)$ ... (1)

These two formulae enable us to find any two of the quantities P, I, A, n and r when the other three are given.

If r denotes the rate per cent. per annum, as is usually the case, then $\frac{r}{100}$ must be substituted for r in the above formulae ; so that they would stand thus :

$$I=\frac{Pnr}{100} \text{ ; and } A=P\left(1+\frac{nr}{100}\right) \quad \cdots \quad \cdots \quad (2)$$

Thus, *in the case of simple interest the amount increases in Arithmetical Progression.*

Example 1. *Find the simple interest and amount of Rs. 5000 for $6\frac{1}{2}$ years at $3\frac{1}{2}$ per cent. per annum.*

The required interest $=$ Rs. $5000 . 6\frac{1}{2} . \frac{3\frac{1}{2}}{100}$

$= \text{Rs. } 5000 \times \frac{13}{2} \times \frac{7}{2} \times \frac{1}{100}$

$= \text{Rs. } 1137\frac{1}{2} = \text{Rs. } 1137.8 \text{ as.}$

Hence the amount $=$ Rs. 5000 + Rs.1137.8 as. $=$ Rs.6137.8as.

Example 2. *What must be the rate of interest so that a sum of money may double itself at the end of 20 years ?*

Let r be the rate of interest per cent. per annum.

Hence, $A=P(1+\frac{nr}{100})$

Here. $n=20$, and $A=2P$, because P doubles itself.

$\therefore 2P= P(1+\frac{20r}{100})$, or $2 = 1+\frac{20r}{100}$, or $1 = \frac{r}{5}$. $\therefore r=5$.

Hence the rate of interest is 5 per cent. per annum.

211. Compound Interest and its Formulae.

If the interest is converted into principal at the end of every year, then we have, with the notation of the previous article.

the amount of P in 1 year $=P(1+r)$

" " 2 years $=$ the amount of $P(1+r)$ in 1 year

$=P(1+r)(1+r)=P(1+r)^2$

" " 3 " $=$ the amount of $P(1+r)^2$ in 1 year

$=P(1+r)^2(1+r)=P(1+r)^3$, etc.

Hence " n " $=P(1+r)^n$.

Thus, $A=P(1+r)^n$, and $I=A-P=P\{(1+r)^n-1\}$ $\cdots$ (1)

If we put $1+r=R$, then the formulæ become

$$A=PR^n, \text{ and } I=P(R^n-1) \cdots\cdots\cdots \quad (2)$$

Thus, *in the case of compound interest the amount increases in Geometrical Progression.*

The interest may be converted into principal at any interval, and not necessarily at the end of a year. But it is evident that the formulæ found above may be easily modified to suit the case when the interval is other than a year. Thus, if the interest is converted into the principal at the end of every pth part of a year, then the amount and interest at the end of n years will be given by

$$A=P\left(1+\frac{r}{p}\right)^{pn} \text{ and } I=P\left\{\left(1+\frac{r}{p}\right)^{pn}-1\right\}\cdots \qquad (3)$$

Thus, when a sum of money P is lent out at compound interest with half-yearly, quarterly, monthly *rests* (*i.e.*, the interest is converted into principal at the end of these intervals), we have, at the end of n years,

$$A=P(1+\tfrac{1}{2}r)^{2n}, \qquad A=P(1+\tfrac{1}{4}r)^{4n}, \qquad A=P(1+\tfrac{1}{12}r)^{12n},$$

respectively.

It is to be noted, however, that usually in business transactions, when the time contains a fraction of the period of rest, simple interest is calculated for this fractional period. Thus, in compound interest with annual rests, the amount at the end of $\left(n+\frac{1}{q}\right)$ years is $P(1+r)^n\left(1+\frac{r}{q}\right)$.

The formulæ for compound interest are very much suited for logarithmic calculation; and problems in compound interest are usually worked out with the help of logarithms.

Thus since $A=P(1+r)^n$, we have, on taking logarithms,

$$\log A=\log P+n\log(1+r) \qquad (4)$$

which is the formula usually adopted in practice.

It is interesting to inquire what happens when the interest is converted into principal continuously. In this case the periods of rest become instantaneous, *i.e.*, reduce to nothing. Hence the amount at the end of n years will be obtained, by making p infinite in the formula

$$A=P\left(1+\frac{r}{p}\right)^{pn}.$$

But from the Exponential Theorem, if p is infinite,

$$\left(1+\frac{r}{p}\right)^{pn}=1+nr+\frac{n^2r^2}{\underline{|2}}+\frac{n^3r^3}{\underline{|3}}+\cdots ad\ inf.=e^{nr}.$$

Therefore, $$A=Pe^{nr} \qquad (5)$$

Taking logarithms, $\log_e A=\log_e P+nr.$ (6)

Example 1. *Find in how many years Rs. 100 will become Rs. 1000 at 4 p. c. compound interest. Give the answer correct to two places of decimals.* (*Given log* $104 = 2{\cdot}0170333$).
[*Cal. Int.* 1934]

The logarithmic formula is $\log A = \log P + n \log(1+r)$.

Here, $A = 1000$, $P = 100$, $r = \frac{4}{100}$.

$\therefore \quad \log 1000 = \log 100 + n \log(1 + \frac{4}{100})$

or, $\quad n \log \frac{104}{100} = \log 1000 - \log 100 = 3 - 2 = 1.$

or, $\quad n(\log 104 - \log 100) = 1$

or, $\quad n(2{\cdot}0170333 - 2) = 1$

or, $\quad n \times {\cdot}0170333 = 1 \quad \therefore\ n = \dfrac{1}{{\cdot}0170333} = 58{\cdot}71$ nearly.

Hence, the required no. of years is $58{\cdot}71$ years.

Example 2. *At what rate per cent. compound interest will* £175 *amount to* £ 192. 18*s*. 9*d*. *in two years* ? [*Cal. F. A.* 1879.]

From the formula $A = P(1+r)^n$, we have, putting A $= £192.18s.9d. = £192\frac{15}{16} = £\frac{3087}{16}$, $P = £175$, $n = 2$,

$$\frac{3087}{16} = 175.\,(1+r)^2$$

$\therefore \quad (1+r)^2 = \frac{3087}{16 \times 175} = \frac{441}{400} = (\frac{21}{20})^2$

or, $\quad 1 + r = \frac{21}{20} \quad \therefore\ r = \frac{1}{20}.$

Hence the rate of interest is 5 per cent. per annum.

213. Present Worth and Discount.

In a case of *simple interest*, if P represents the present worth of an *amount* A, due n years hence, we have evidently

$$A = P(1 + nr)$$

whence $\quad P = \dfrac{A}{1+nr} \quad \cdots \quad \cdots \quad (1)$

Hence discount $D = A - \dfrac{A}{1+nr} = \dfrac{Anr}{1+nr} \quad \cdots \quad \cdots \quad (2)$

If r denotes the rate of interest per cent. per annum, we have as before to substitute $\frac{r}{100}$ for r in these formulæ, so that these become

$$P = \frac{100A}{100+nr}, \text{ and } D = \frac{Anr}{100+nr} \quad \cdots \quad \cdots \quad (3)$$

In a case of *compound interest*, we have, with the same notation as above,

$$A = P(1+r)^n \text{ or } PR^n$$

whence $\quad P = A(1+r)^{-n}$ or $AR^{-n} \quad \cdots \quad (4)$

Hence $\quad D = A - A(1+r)^{-n}$ or $A(1 - R^{-n}) \quad \cdots \quad (5)$

Example 1. *What is the present worth of Rs.* 1000 *payable* 2 *years hence, at* 5 *per cent. per. annum compound interest ?*

In the formula $\log P = \log A - n \log (1+r)$, putting $A = 1000$, $n=2$, $r=\frac{5}{100}$, we have $\log P = \log 1000 - 2 \log (1+\frac{5}{100})$

$$\begin{aligned} &= 3 - 2 \log 105 + 2 \log 100 \\ &= 3 - 2 \times 2{\cdot}0211893 + 4 \\ &= 7 - 4{\cdot}0423786 \\ &= 2{\cdot}9576214 \\ &= \log 907, \text{ nearly.} \end{aligned}$$

Hence the present worth is Rs. 907 nearly, and therefore, the discount is Rs. 93 nearly.

214. Present value of an Annuity.

(*a*) *To find the present value of an annuity A payable at the end of each of* n *successive years, at compound interest.*

With the same notation as before, we have

the present value of the first payment $= A(1+r)^{-1}$

" second " $= A(1+r)^{-2}$

" third " $= A(1+r)^{-3}$

" nth " $= A(1+r)^{-n}$

Hence the present value of the whole annuity for n years

$$\left\{\frac{1}{1+r} + \frac{1}{(1+r)^2} + \cdots + \frac{1}{(1+r)^n}\right\}$$

$$= \frac{A}{r}\left\{1 - \frac{1}{(1+r)^n}\right\} \text{ or } \frac{A}{R-1}[1-R^{-n}] \qquad (1)$$

(*b*) *To find the present value of a perpetual annuity.*

If the annuity is perpetual, *i.e.*, if the payment is to be continued for ever, its present value can be found by putting $n=\infty$ in the above formula. Thus, we have the present value of a perpetual annuity A

$$= \frac{A}{r}\left\{1 - \frac{1}{(1+r)^n}\right\}\left(\text{where } n=\infty\right) = \frac{A}{r}. \qquad (2)$$

If the present value of an annuity for any number of years p be pA, then the annuity is said to be worth p *years' purchase.*

If the annuity is perpetual, we have $pA = \dfrac{A}{r}$.

Hence, $$p = \frac{1}{r} = \frac{100}{\text{rate per cent.}} \qquad (1)$$

That is to say, *the number of years' purchase of a perpetual annuity is obtainable by dividing* 100 *by the rate per cent.*

This is the formula by which the sales and purchases of *Freehold Estates* are regulated.

All annuities which take effect immediately, and which are termed *annuities certain* or *annuities in possession*, are covered by the formulæ of this article.

Example 1. *Find the present value of an annuity of Rs.* 300 *per annum, for* 5 *years at* 4 *per cent.* (*Given log* 1·04 = ·0170333, *and* log 821·923 = 2·9148335.) [*Cal. Int.* 1936]

In the formula, Present value $=\frac{A}{r}\left\{1-\frac{1}{(1+r)^n}\right\}$, putting $A=300$, $r=\frac{4}{100}$, and $n=5$, we have

$$\text{the present value}=300.\frac{100}{4}\left\{1-\frac{1}{(1+\frac{4}{100})^5}\right\}$$
$$=\frac{30000}{4}\{1-(\tfrac{100}{104})^5\}.$$

New, to find the value of $(\frac{100}{104})^5$, let it be x.

Then, $\log x=5(\log 100-\log 104)=5(2-2{\cdot}0170333)$
$=-5\times{\cdot}0170333=-{\cdot}0851665$
$=\bar{1}{\cdot}9148335=\log{\cdot}821923$

$\therefore\ x={\cdot}821923.$

Hence present value $=\frac{30000}{4}(1-{\cdot}821923)$
$=\frac{30000}{4}\times{\cdot}178077$
$=1335{\cdot}58$ nearly.
$=$ Rs. 1335. 9as. 2 8 p. nearly.

If this annuity be perpetual, then

$$\text{the present value}=\frac{A}{r}=\frac{300}{\frac{4}{100}}=\frac{300\times100}{4}$$
$$=\frac{30000}{4}=\text{Rs. }7500.$$

Example 2. *How many years' purchase should be given for a free-hold estate, when money is worth* $3\frac{1}{2}$ *per cent ?*

In the formula $p=\frac{1}{r}$, putting $r=\frac{3\frac{1}{2}}{100}=\frac{7}{200}$, we have the required no. of years' purchase

$$p=\frac{1}{r}=\frac{200}{7}=28\tfrac{4}{7}\text{ years.}$$

215. Present Value of Deferred Annuity

(*a*) *To find the present value of an annuity A which is to take effect at the end of m years and then to continue for n years.*

With the same notation as before, we have the present value of the first, second, third, ... payments to be made at the end of $m+1$, $m+2$, $m+3$, ... years repectively, equal to

$$A(1+r)^{-(m+1)},\ A(1+r)^{-(m+2)},\ A(1+r)^{-(m+3)}\ldots$$

or $$AR^{-(m+1)},\ AR^{-(m+2)},\ AR^{-(m+3)},\ldots \text{where } R=1+r.$$

Hence the present value of the whole annuity for n years deferred m years,

$$=A\left\{\frac{1}{R^{m+1}}+\frac{1}{R^{m+2}}+\cdots\cdots+\frac{1}{R^{m+n}}\right\}=\frac{A}{R^{m+1}}\cdot\frac{1-\frac{1}{R^n}}{1-\frac{1}{R}}$$

$$=\frac{A}{R^{m+1}}\cdot\frac{R^n-1}{R-1}\cdot\frac{1}{R^{n-1}}=\frac{A}{R^{m+n}}\cdot\frac{R^n-1}{R-1} \qquad \cdots \qquad (1)$$

$$=\frac{A}{R-1}\left\{\left(1-\frac{1}{R^{m+n}}\right)-\left(1-\frac{1}{R^m}\right)\right\} \qquad \cdots \qquad (2)$$

= the present value of the annuity for $m+n$ years
− the present value for m years.

(*b*) *To find the present value of a perpetual annuity deferred m years.*

If in the above formula we put $n=\infty$, we get the present value of a perpetual annuity, deferred m years,

$$=\frac{A}{R-1}\cdot\frac{1}{R^m} \qquad \left[\text{since } \frac{1}{R^{m+n}}=0, \text{ when } n=\infty\right]$$

$$=\frac{A}{R-1}R^{-m} \text{ or } \frac{A}{r}(1+r)^{-m} \qquad \cdots \qquad \cdots \qquad \cdots \qquad (3)$$

These formulæ enable us to calculate the value of *Annuities in Reversion*; and the last formula determines the value of the *Absolute Reversion* of an annuity or of the *fee simple* of a free-hold Estate which is to fall in at the expiration of m years.

Example 1. *Find the present value of a perpetual annuity of £729, the first payment of which is to be made in three years' time, interest being reckoned at* 8 *per cent.* [*Cal. F. A.* 1881]

Since the first payment is to be made in three years' time, the annuity is really deferred only two years. Hence in the formula, Present value $=\frac{A}{r}(1+r)^{-m}$,

putting $A=729$, $r=\frac{8}{100}$, $m=2$, we have

the present value $=729\cdot\frac{100}{8}\left(1+\frac{8}{100}\right)^{-2}=729\cdot\frac{100}{8}\left(\frac{100}{108}\right)^2=\frac{15625}{2}$
$=7812\frac{1}{2}=£7812.\ 10s.$

216. Amount of an Annuity unpaid for a number of years.

If it is a case of *simple interest*, let A be the annuity, and n the number of years for which the annuities are not paid.

The first payment which was due at the end of the first year has therefore been deferred for $n-1$ years. Hence, the amount corresponding to the 1st payment $= A\{1+(n-1)r\}$.

Similarly, amount for 2nd payment $= A\{1+(n-2)r\}$

" " 3rd " $= A\{1+(n-3)r\}$

and so on.

Therefore the whole amount due at the end of n years

$$= A\{1+(n-1)r\} + A\{1+(n-2)r\} + \ldots\ldots \text{to } n \text{ terms}$$
$$= nA + Ar\{1+2+3+\cdots\cdots(n-1)\}$$
$$= nA + \tfrac{1}{2}n(n-1).Ar$$
$$= \{n + \tfrac{1}{2}n(n-1)r\}A \qquad \cdots \qquad \cdots \qquad \cdots \qquad (1)$$

If it is a case of *compound interest*, we have, with the same notation as before,

the amount corresponding to the 1st payment $= A(1+r)^{n-1}$

" " 2nd " $= A(1+r)^{n-2}$

" " 3rd " $= A(1+r)^{n-3}$

and so on.

Therefore, the whole amount due at the end of n years

$$= A\{(1+r)^{n-1} + (1+r)^{n-2} + \cdots + (1+r) + 1\}$$
$$= A\frac{(1+r)^n - 1}{1+r-1}$$
$$= \frac{A}{r}\{(1+r)^n - 1\} \text{ or } \frac{A}{R-1}(R^n - 1).$$

Example 1. *What will an annuity of Rs.* 2500 *amount to in* 7 *years, paid half-yearly, showing* 6 *per cent. per annum simple interest ?*

In the formula, Amount $= \{n + \frac{1}{2}n(n-1)r\}A$, putting $A = 1250$, $n = 14$, $r = \frac{3}{100}$, since the annuity is paid half-yearly so that the time is 14 half-years, and rate of interest is 3 per cent. per half-year, we have

$$\text{Amount} = \{14 + \tfrac{1}{2}.\,14.\,13.\,\tfrac{3}{100}\}1250 = \{14 + \tfrac{273}{100}\}1250$$
$$= 17500 + 3412\tfrac{1}{2} = 20912\tfrac{1}{2} = \text{Rs. } 20912.\ 8a.$$

Example 2. *How much will an annuity of Rs.* 500 *amount to in* 20 *years at* $3\frac{1}{2}$ *per cent. compound interest?* (*Given* $\log 1035 = 3{\cdot}0149403$ *and* $\log 1{\cdot}989784 = {\cdot}2988060$.)

In the formula, $\text{Amount} = \frac{A}{r}\{(1+r)^n - 1\}$, putting $A = 500$,

$x = \frac{3\frac{1}{2}}{100} = \frac{7}{200}$, $n = 20$, we have

$$\text{Amount} = 500 . \tfrac{200}{7}\{(1 + \tfrac{7}{200})^{20} - 1\}$$
$$= \tfrac{100000}{7}\{(\tfrac{207}{200})^{20} - 1\}.$$

We have to find out the value of $(\frac{207}{200})^{20}$. Let it be x.

Then $$x = (\tfrac{207}{200})^{20} = (\tfrac{1035}{1000})^{20}$$

Taking logarithms, $\log x = 20\,[\log 1035 - \log 1000]$
$$= 20[3{\cdot}0149403 - 3] = 20 \times {\cdot}0149403$$
$$= {\cdot}2988060 = \log 1{\cdot}989784$$

$\therefore$ $$x = 1{\cdot}989784.$$

$\therefore$ $$\text{Amount} = \tfrac{100000}{7} \times {\cdot}989784$$
$$= \tfrac{989784}{70} = 14139{\cdot}8 \text{ nearly}.$$
$$= \text{Rs. } 14139.\ 12\ as.\ 9{\cdot}6\ p. \text{ nearly}.$$

EXERCISE 71.

1. If Rs. 1258. 5*a*. 4*p*. is lent out at 5 per cent. per annum, find the simple interest and amount at the end of 4 years.

2. The sum of Rs. 250 amounted to Rs. 289. 6*a*. in $3\frac{1}{2}$ years at simple interest; what was the rate of interest?

3. What sum must be paid down to receive Rs. 600 at the end of 9 months, allowing 5 p. c. per annum simple interest?

4. The interest on a certain sum of money for 3 years is Rs. 825 and the discount for the same time is Rs. 645, simple interest being reckoned in both cases. Find the rate per cent. per annum and the sum.

5. *A* sells goods to *B* and allows him 10 p. c. discount, if he pay in 6 months; what discount should he allow if payment be made in 2 months, at 5 p. c. per annum simple interest?

6. Show that the discount is half the Harmonic Mean between the sum due and the simple interest on it.

7. Find the compound interest and amount of Rs. 160 in 4 years at 6 p. c. per annum.

8. In how many years will a sum of money double itself at 5 p. c. compound interest ? [Cal. Int. 1938]

9. A man invests £30 a year in a Savings Bank which pays $2\frac{1}{2}$ p. c. per annum on all deposits. What will be the total amount at the end of 20 years ? [Cal. Int. 1937]

10. Show that Rs. 1000 will become Rs. 2500 at 10 p. c. per annum compound interest in about 9·6 years. [Given $\log 2 = 0·3010300$, $\log 11 = 1·0413927$.] [Cal. Int. 1940]

11. If an annuity of Rs. 700 be forborne 5 years, what will it amount to at 5 p. c. simple interest ?

12. If an annuity of Rs. 300 be forborne 16 years, what will it amount to at $4\frac{1}{2}$ p. c. compound interest ?

13. What sum should be paid for an annuity of £100 a year to be paid for 40 years, money being supposed to be worth 4 p. c. per annum ? [Cal. Int. 1935]

14. Find the present value of an annuity of Rs. 2400 for 20 years at $4\frac{1}{2}$ p. c. compound interest per annum [Given $\log 1·045 = 0·0191$ and $\log 4·150 = 0·6180$.] [Cal. Int. 1939]

15. Find the present value of an annuity of Rs. 10000 to continue for 25 years, allowing compound interest at 5 p.c. per annum. [Given $\log 105 = 2·0211893$, $\log 2·95862 = ·4702675$] [Cal. Int. 1942]

16. Find the amount of an annuity of Rs. 100 in 20 years allowing compound interest at $4\frac{1}{2}$ p. c. [Given $\log 1·045 = ·0191163$, $\log 24·117 = 1·3823260$.] [Cal. Int. 1941]

17. Find the present value of an annuity of £100 for 14 years to commence at the expiration of 7 years, allowing 5 p. c. compound interest.

18. Find the present value of a perpetual annuity of £75 allowing 3 p. c. compound interest.

19. If the annual rental of a free-hold estate be £ A, what is its present worth at 4 p. c. ?

20. What is the present value of an estate of Rs. 1000 a year at 5 p. c. (i) to be entered on immediately, and (ii) after 8 years ?

21. What is the present value of the reversion of £100 per annum for 8 years and to commence at the expiration of two years, allowing 4 p. c. compound interest? [Given log 1·04 = 0·170333, log 6·5564 = ·8296670 and log 9·2 = ·9659334.]

22. If an estate in fee be sold for Rs. 7250 in ready money, what is the yearly rent, allowing 4 p. c. to the buyer?

23. A corporation borrows £30000 which is to be repaid by 30 equal yearly payments. How much will have to be paid each year money being supposed to be worth 4 p. c. per annum?

24. How many years' purchase should be given for a freehold estate at 5 p. c. compound interest?

25. If a, b, c years' purchase must be paid for an annuity to continue for n, $2n$, $3n$ years respectively, show that $a^2 - ab + b^2 = ac$.

ANSWERS

Exercise 1. [Pages 13–15]

1. $\frac{3}{x^{\frac{2}{5}}}$; $\frac{5}{x^{\frac{1}{3}}a^{\frac{2}{3}}}$; $\frac{x^{\frac{2}{3}}}{3}$; $\frac{a^{\frac{4}{3}}}{x^{\frac{3}{4}}}$; $\frac{1}{ax}$

2. $\sqrt[4]{a^3}$; $\frac{1}{\sqrt[4]{x}}$; $\frac{1}{\sqrt[5]{x^3}}$; $\frac{2}{\sqrt[x]{a}\cdot\sqrt[y]{b}}$; $\frac{\sqrt[4]{y^3}}{\sqrt[3]{x^2}}$; $\frac{\sqrt[x]{b^y}}{\sqrt[y]{a^x}}$.

3. $\frac{1}{a^{\frac{1}{3}}}$; $a^{\frac{3}{2}}$; a; $\frac{1}{x^{\frac{1}{2}}}$; x^{m^n}; a^2.

4. 8; $\frac{1}{2}$; $\frac{1}{32}$; 81, $\frac{1}{125}$; 27; 27; $\frac{16}{625}$.

5. (i) $x^{\frac{1}{3}}$. (ii) x. (iii) $\frac{1}{ab^{\frac{1}{9}}}$. (iv) $x^{\frac{1}{12}}$.

(v) $\frac{a^{\frac{1}{3}}}{b^{\frac{3}{8}}}$. (vi) $\frac{1}{y^{\frac{1}{3}}}$. (vii) x^a. (viii) $\frac{1}{x^m}$.

(ix) x^{a+b+c}. (x) 1. (xi) x. (xii) $a^{\frac{1}{m}}$.

(xiii) $\frac{1}{x^{p^2+q^2}}$. (xiv) 1. (xv) $\left(\frac{p}{q}\right)^{2m}$. (xvi) $\left(\frac{p}{q}\right)^{p+q}$.

(xvii) $\frac{a}{b}$. (xviii) $\left(\frac{y}{x}\right)^{\frac{1}{3}}$. (xix) 1. (xx) 1

6. $x^{2^n}-y^{2^n}$.

7. $x^{2^n}+a^{2^{n-1}}x^{2^{n-1}}+a^{2^n}$.

8. $x^{2^{n-1}}+a^{2^{n-2}}x^{2^{n-2}}+a^{2^{n-1}}$.

9. $a^{2\cdot 3^{n-1}}+b^{2\cdot 3^{n-1}}+c^{2\cdot 3^{n-1}}-b^{3^{n-1}}c^{3^{n-1}}-c^{3^{n-1}}a^{3^{n-1}}$
$-a^{3^{n-1}}b^{3^{n-1}}$

11. 0.

12. $x-2x^{\frac{1}{2}}+3-2x^{-\frac{1}{2}}+x^{-1}$.

13. $x^{\frac{4}{3}}y^{-\frac{4}{3}}+x^{\frac{2}{3}}y^{-\frac{2}{3}}+x^{-\frac{2}{3}}y^{\frac{2}{3}}+x^{-\frac{4}{3}}y^{\frac{4}{3}}$.

14. $a^{\frac{2}{3}}b^{-\frac{2}{3}}+b^{\frac{2}{3}}c^{-\frac{2}{3}}+c^{\frac{2}{3}}a^{-\frac{2}{3}}-a^{-\frac{1}{3}}b^{\frac{1}{3}}-b^{-\frac{1}{3}}c^{\frac{1}{3}}-c^{-\frac{1}{3}}a^{\frac{1}{3}}$.

Exercise 2. [Page 17.]

1. $x=\frac{1}{3}$. 2. $x=-\frac{3}{2}$. 3. $x=2$. 4. $x=2$.

5. $x=-\frac{b}{a}$. 6. $x=0, \frac{2}{3}$. 7. $x=1$. 8. $x=\frac{1}{2}$.

9. $x=\frac{203}{46}$. 10. $x=\frac{2}{3}$. 11. $x=\left(\frac{m}{2}\right)^{\frac{1}{m-2}}$ 12. $x=2$.

13. $x=\frac{p-q}{p+q}$. 14. $x=4, y=2$. 15. $x=\left(\frac{a}{b}\right)^{\frac{a}{a-b}}, \ y=\left(\frac{a}{b}\right)^{\frac{b}{a-b}}$.

16. $x=y=a$; or $x=\frac{1}{a}, \ y=-\frac{1}{a}$. 17. $x=2, y=1$.

18. $x=2, y=1$. 19. $x=3, y=2$. 20. $x=1, y=1$.

21. $x=3, y=3$. 22. $x=1, y=\frac{1}{2}, z=1$. 23. $x=y=z=\frac{1}{3}a$.

24. $x=y=z=a$. 27. $m=n^{\frac{1}{n-1}}$ 28. $n=x^{m^{-\frac{1}{x}}}$.

Exercise 3. [Pages 21 – 22.]

1. $3x-4y$. 2. $a-b+c$. 3. x^2+2x+3.

4. $2x-3y+4$. 5. x^3+4x-1. 6. $2x^2+2xa+4b^2$.

7. $1-2x+3x^2-4x^3$. 8. x^3-3x^2+x-4. 9. $\frac{x}{a}-1+\frac{a}{x}$.

x^2 $\frac{a}{3}$ 11. $a-1+$ 12. x^2 $\frac{2x}{3}-\frac{3}{4}$.

13. $\frac{x}{y}-\frac{1}{2}-\frac{y}{x}$. 14. $\frac{2x}{7y}+\frac{3y}{4x}-5$. 15. $\frac{a^2}{b^2}-\frac{a}{b}+\frac{b}{a}-\frac{b^2}{a^2}$

16. 1. 17. a^3-b^3. 18. $x^{\frac{3}{4}}+3x^{-\frac{1}{4}}+2$.

19. $a^{\frac{1}{2}}+b^{-\frac{1}{2}}+a^{-\frac{1}{2}}b^{\frac{1}{2}}$. 20 $a^{\frac{2m}{3n}}+a^{\frac{m}{3n}}-1$. 21. $2a^{\frac{1}{2}}-3b^{\frac{1}{3}}+4c^{\frac{1}{4}}$.

22. $y^{\frac{1}{3}}(x^{\frac{1}{3}}-y^{\frac{1}{3}})+x^{\frac{2}{3}}y^{-\frac{1}{2}}$. 23. $3x^{\frac{2}{3}}+2x^{-\frac{1}{3}}-\sqrt{2}.x^{-\frac{1}{2}}$.

24. (i) $1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3$. (ii) $1-\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3$.

(iii) $1-x-\frac{1}{2}x^2-\frac{1}{2}x^3$. (iv) $a^{\frac{1}{2}}+\frac{x}{2a^{\frac{1}{2}}}-\frac{x^2}{8a^{\frac{3}{2}}}+\frac{x^3}{16a^{\frac{5}{2}}}$.

(v) $1+\frac{1}{2}x+\frac{3}{8}x^2-\frac{3}{16}x^3$. (vi) $1-\frac{1}{2}x-\frac{5}{8}x^2-\frac{5}{16}x^3$.

Exercise 4. [Pages 24—25.]

1. $a-b+c$. 2. $2x^2+3$. 3. $2+\frac{1}{2}b-c$.
4. a^2+b^2. 5. $a-b+c-d$. 6. $a^2-b^2+c^2-d^2$.
7. $x^2+9x+19$. 8. x^2-3x+1. 9. x^2+4x+5.
10. $x^2+5ax+5a^2$. 11. $\frac{x}{y}-1+\frac{y}{x}$. 12. $x-2-\frac{1}{x}$.
13. $x^2-2+\frac{1}{x^2}$. 14. $x^2+2-\frac{1}{x^2}$. 15. $\frac{a}{bc}+\frac{b}{ca}+\frac{c}{ab}$.
16. $\frac{x^2}{2y^2}+\frac{2y^2}{x^2}+1$. 17. $3a^{\frac{1}{2}}-2-a^{-\frac{1}{2}}$. 18. $\frac{a^2+2ab-b^2}{a^2-b^2}$.
19. $x^3+\frac{1}{x^3}+3\left(x+\frac{1}{x}\right)$. 20. $\frac{x^2}{y^2}+\frac{y^2}{x^2}-\left(\frac{x}{y}+\frac{y}{x}\right)+1$.
21. $(a-b)(ab+1)$. 22. $\frac{2x}{3y}-\frac{3y}{4z}-\frac{4x}{5z}$.
23. $a^2+\frac{1}{a^2}+a-\frac{1}{a}$. 24. $\frac{m(a-x)^2+n(a+x)^2}{a^2-x^2}$.
25. $x-\frac{1}{x+2}$. 26. $p^2=4q$. 27. $a=24$. 28. $a=4$.
29. $2x^2+2x+1$; 221. 30. $(ac-bd+bc+ad)^2$.

Exercise 5. [Pages 29—30.]

1. (i) $\sqrt[3]{64}$. (ii) $\sqrt[4]{625}$. (iii) $\sqrt[6]{\frac{1}{64}}$. (iv) $\sqrt[4]{(a+b)^2}$
2. (i) $\sqrt{48}$. (ii) $\sqrt{275}$. (iii) $\sqrt[3]{40}$. (iv) $\sqrt[4]{7203}$.
(v) $\sqrt[n]{a^{2n}b^3}$. (vi) $\sqrt[3]{x^6y}$. (vii) $\sqrt[7]{a^{21}b^2}$. (viii) $\sqrt[n]{a^{2n}b^{3n}c}$.
3. (i) $6\sqrt{2}$. (ii) $7\sqrt{2}$. (iii) $-11.\sqrt[3]{7}$. (iv) $ab.\sqrt[3]{a^2b}$.
(v) $2a^2b.\sqrt[3]{3b^2}$. (vi) $3yz^2.\sqrt[4]{2x^3yz}$.
4. (i) $\sqrt[6]{8}$, $\sqrt[6]{9}$. (ii) $\sqrt[12]{16}$, $\sqrt[12]{27}$. (iii) $\sqrt[12]{16}$, $\sqrt[12]{27}$, $\sqrt[12]{625}$.
(iv) $\sqrt[30]{14348907}$, $\sqrt[30]{1048576}$, $\sqrt[30]{64}$.
(v) $\sqrt[6]{a^3}$, $\sqrt[6]{b^4}$, $\sqrt[6]{c}$. (vi) $\sqrt[30]{x^{10}}$, $\sqrt[30]{y^{18}}$, $\sqrt[30]{z^{21}}$.
5. (i) $\sqrt{3}$. (ii) $\sqrt[3]{19}$. (iii) $\sqrt[3]{3}$. (iv) $\sqrt[4]{17}$. (v) $\sqrt[3]{4}$. (vi) $\sqrt[4]{19}$.
6. (i) $\sqrt{2}$, $\sqrt[4]{7}$, $\sqrt[3]{5}$. (ii) $\sqrt[4]{13}$, $\sqrt[6]{47}$; $\sqrt[3]{7}$.
(iii) $\sqrt[3]{2}$, $\sqrt[6]{5}$. $\sqrt[4]{3}$. (iv) $\sqrt[4]{82}$, $\sqrt[3]{46}$, $\sqrt{13}$.
7. (i) $2\sqrt{3}$, $5\sqrt{3}$, $8\sqrt{3}$. (ii) $3\sqrt{2}$, $4\sqrt{2}$, $7\sqrt{2}$, $6\sqrt{2}$
(iii) $3.\sqrt[3]{2}$, $5.\sqrt[3]{2}$, $6.\sqrt[3]{2}$. (iv) $2.\sqrt[3]{3}$, $3.\sqrt[3]{3}$, $5.\sqrt[3]{3}$.
(v) $bc\sqrt{a}$, $b^2c^3\sqrt{a}$, $(b+c)\sqrt{a}$.

8. (i) $5\sqrt{2}, 3\sqrt{5}$. (ii) $2\sqrt{5}, 4\sqrt{3}, 4\sqrt{2}$.
(iii) $11\sqrt{2}, 7\sqrt{3}, 6\sqrt{5}$. (iv) $2.\sqrt[3]{7}, 2.\sqrt[3]{4}, 3.\sqrt[3]{3}$.

Exercise 6. [Page 31.]

1. $12\sqrt{2}$. **2.** $\sqrt{5}$. **3.** $5.\sqrt[3]{5}$. **4.** $\sqrt[3]{4}$.
5. $4\sqrt{2}$. **6.** 0. **7.** $(a+b+c)\sqrt{ab}$. **8.** $2xyz.\sqrt[3]{a}$.

Exercise 7. [Page 31.]

1. $7\sqrt{6}$. **2.** $6\sqrt{15}$. **3.** 30. **4.** $8\sqrt{6}$.
5. $x\sqrt{6ab}$. **6.** $10x.\sqrt[3]{3}$. **7.** $24.\sqrt[6]{2}$. **8.** $9.\sqrt[12]{3}$.
9. $10.\sqrt[6]{500}$. **10.** $3.\sqrt[6]{3888}$. **11.** $12.\sqrt[3]{4}$. **12.** $15.\sqrt[6]{36000}$.

Exercise 8. [Pages 32–33.]

1. $2\sqrt{3}+3\sqrt{2}$. **2.** $5\sqrt{3}-3\sqrt{5}$. **3.** $72+23\sqrt{6}$.
4. $-(42+\sqrt{15})$. **5.** $6\sqrt{2}+6\sqrt{3}-3\sqrt{6}-6$. **6.** $6\sqrt{3}+3\sqrt{30}$.
7. $1+5\sqrt{35}$. **8.** $1+\sqrt{2}$. **9.** 8.
10. 1. **11.** $a-b$. **12.** $6a-20b+7\sqrt{ab}$. **13.** 1. **14.** x.
15. 4. **16.** $4+2\sqrt{10}$.
17. $2(b+\sqrt{ab})$. **18.** $7\sqrt{2}+5\sqrt{3}-\sqrt{6}-12$.
19. $15-5\sqrt{6}+10\sqrt{3}-10\sqrt{2}$. **20.** $7-4\sqrt{3}$
21. $7+2\sqrt{10}$. **22.** $255-40\sqrt{35}$. **23.** $2x-2\sqrt{x^2-1}$.
24. $5x-4-2\sqrt{6x^2-7x-5}$.

Exercise 9. [Pages 35—36.]

1. (i) $\frac{2}{3}$. (ii) $\frac{1}{3}.\sqrt[6]{2}$. (iii) $\frac{1}{6}.\sqrt[6]{36000}$.
(iv) $\frac{1}{2}.\sqrt[6]{500}$. (v) $\frac{4}{15}.\sqrt[6]{1125}$. (vi) $\frac{1}{3}.\sqrt[12]{243}$.
2. (i) $\frac{1}{6}(3+\sqrt{3})$ or 1·789 (ii) $\sqrt{5}-\sqrt{3}$ or ·504.
(iii) $\frac{1}{30}(114-41\sqrt{6})$ or ·452. (iv) $3-2\sqrt{2}$ or ·172.
3. $\sqrt[3]{4}.x^2-(2+\sqrt[3]{2})x+(2.\sqrt[3]{2}-\sqrt[3]{4}+1)$.
4. (i) $\frac{1}{2}(5\sqrt{2}+4\sqrt{3})$. (ii) $3\sqrt{5}-2\sqrt{11}$. (iii) $5+2\sqrt{6}$.
(iv) $\frac{1}{4}(2+\sqrt{2}+\sqrt{6})$. (v) $\dfrac{1-\sqrt{1-x^4}}{x^2}$.
(vi) $\dfrac{x^2+x+1+(x+1)\sqrt{x^2+1}}{x}$. (vii) $\dfrac{x+\sqrt{x^2-y^2}}{y}$.
(viii) $\dfrac{(\sqrt{b}+\sqrt{c}-\sqrt{a})(\sqrt{c}+\sqrt{a}-\sqrt{b})(\sqrt{a}+\sqrt{b}-\sqrt{c})}{2bc+2ca+2ab-a^2-b^2-c^2}$.

5. (i) 4. (ii) 7. (iii) $2x$. (iv) $\frac{2\sqrt{a^2-x^2}}{x^2}$.

(v) $2\sqrt{3}$. (vi) $2x^2$. **6.** ·82. **7.** $\frac{1}{3}\sqrt{3}$.

Exercise 10. [Page 41.]

1. $2+\sqrt{3}$. **2.** $3-\sqrt{2}$. **3.** $\sqrt{5}+\sqrt{3}$.

4. $2-\sqrt{5}$. **5.** $\frac{1}{\sqrt{2}}(1+\sqrt{11})$. **6.** $3\sqrt{3}-1$.

7. $\sqrt[4]{\frac{3}{4}}(1+\sqrt{5})$. **8.** $3-\sqrt{3}$. **9.** $1+\frac{1}{2}\sqrt{5}$.

10. $2+\frac{1}{5}\sqrt{5}$. **11.** $\sqrt{\frac{a+b}{2}}+\sqrt{\frac{a-b}{2}}$.

12. $a+\sqrt{x^2-a^2}$. **13.** $x-1+\sqrt{2x-1}$. **14.** $b+\sqrt{ab-b^2}$.

15. $x\sqrt{x}+a\sqrt{a}$. **16.** $\sqrt{(2x+1)}+\sqrt{(x+2)}$.

17. $\sqrt{x}+\sqrt{y+z}$. **18.** $\frac{1}{\sqrt{2}}(\sqrt{a+b-c}+\sqrt{a-b+c})$.

19. $1+\sqrt{2}+\sqrt{3}$. **20.** $1+\sqrt{2}-\sqrt{5}$.

Exercise 11. [Pages 44–45.]

1. $\frac{\sqrt{2}+\sqrt{3}+\sqrt{5}}{2}$. **2.** $\frac{a\sqrt{b}+b\sqrt{a}-\sqrt{ab(a+b)}}{2ab}$.

3. $(3-\sqrt{2})(\sqrt{2}-\sqrt[4]{2}+1)$. **4.** $\sqrt{2}$. **6.** $\frac{1}{2}(a-1)$. **7.** 2702.

8. $\frac{3}{2}$. **9.** $\frac{1}{3}\sqrt{3}$. **10.** $\frac{4a^2-1}{4a^2-3}$. **11.** $n(n-1)$.

12. (i) $\frac{1}{\sqrt{2}}\{\sqrt{(1+m)}+\sqrt{(1-m)}\}$.

(ii) $a-\sqrt{ax-a^2}$. (iii) $\sqrt{x+z}+\sqrt{y+w}$.

13. $\frac{1}{2}\left(ab+\frac{1}{ab}\right)$. **17.** $a^2+b^2+c^2-2bc-2ca-2ab=0$.

22. $\frac{x+\sqrt{4x^2+3}}{3\sqrt{x^2+1}}$. **23.** $\sqrt{a+x}$. **24.** $\frac{2(1+ax)}{a+x}$. **25.** $1-a^2$.

Exercise 12. [Pages 51—52.]

1. 3 : 4. **2.** 6 : 11. **3.** 11 : 13.

4. 7 : 11. **5.** $x+3y : x+2y$. **6.** $a+b : b$.

7. $a+b : a-b$. **8.** $x-2y : x-y$. **9.** 4 : 7.

10. 7 : 20. **11.** 1 : 1. **12.** $a^2-b^2 : a^2+b^2$.
13. $(a+b)^2 : a^2+b^2$. **14.** 13. **15.** 8.
16. 27, 48. **17.** 25 : 39. **18.** 15, 33.
19. 7. **20.** 2. **21.** 33, 44.
22. $\frac{a+2b}{b-2a}$. **23.** 1 : 1. **29.** $\frac{ad-bc}{c-a}$. **31.** $\frac{ab}{3b-2a}$.

Exercise 13. [Page 56.]

1. (i) 25. (ii) 49. (iii) 15. (iv) $\frac{1}{6}$. (v) 5. (vi) $a^6b^2c^7$.
2. (i) $4\frac{1}{2}$. (ii) $12\frac{1}{2}$. (iii) 16. (iv) 36. (v) $\frac{2}{5}$. (vi) $a+b$.
3. (i) 15. (ii) 12. (iii) 30. (iv) $\frac{3}{2}$. (v) $6xyz$. (vi) $a^{\frac{3}{2}}b^{\frac{3}{2}}$.
4. $\frac{24}{5}$. **8.** 3.

Exercise 15. [Pages 65—67.]

1. 0. **16.** 0. **22.** $ad=bc$. **31.** A 72 yrs. B 63 yrs.
32. In favour of the second army.

Exercise 16. [Pages 74—76.]

1. $y=28$. **2.** $8y=13x$. **3.** $\frac{ab}{c}$. **4.** 2. **5.** 9. **6.** $\frac{ab}{c}$.
7. 6. **8.** $y=2x+\sqrt{x}$. **9.** $y=3x^2-7x+2$; 2, $\frac{1}{3}$.
10. $\frac{1}{p}$, ab. **23.** $x=-\frac{2}{15}\left(11x+\frac{1}{z}\right)$. **29.** $mn+nl+lm+2lmn=1$, where l, m, n are the constants of variation.

Exercise 17. [Pages 78—80.]

1. $5\frac{2}{5}$ days. **2.** 5 days. **3.** 5 days. 4. 13 inches.
5. $(r^3+r'^3)^{\frac{1}{3}}$. **7.** 3·3 secs. **8.** 250 in. **9.** 3·5 secs. nearly.
10. $240m^2p : n^2q$. **11.** 9 : 4. **12.** 4. **13.** $1\frac{5}{9}$ ft.
14. (i) 128 lbs. ; (ii) $199\frac{17}{20}$ lbs. **15.** $64\frac{5}{16}$ lbs. **16.** 18 sq. ft.
17. 72 students. **18.** 20 miles per hour.
19. £ 9. 7*s*. 6*d*., when velocity is 12 miles per hour. **20.** 112.

Exercise 18. [Pages 83—84.]

1. $\pm\sqrt{2}$. **2.** ± 6. **3.** ± 4. **4.** ± 1.
5. $\pm\sqrt{a^2-2a}$. **6.** $\pm\frac{1}{2}\sqrt{23}$. **7.** $\pm\frac{25}{3}$. **8.** ± 1.

9. $\pm 2\sqrt{2}$. 10. $\pm\sqrt{a^2-b^2}$. 11. ± 1. 12. $\pm\frac{1}{3}\sqrt{57}$.
13. $\pm\sqrt{ab}$. 14. ± 2. 15. $\pm\frac{1}{2}\sqrt{26}$. 16. $\pm\sqrt{3}$.
17. $\pm\sqrt{3}$. 18. $\pm\frac{1}{2}\sqrt{5}$. 19. ± 5.
20. $\pm\sqrt{\dfrac{2ab}{a^2+b^2}}$. 21. $\pm\dfrac{b}{2\sqrt{a^2-ab}}$. 22. $\pm\dfrac{2}{a^2}\sqrt{a^2-1}$.
23. $\pm\dfrac{ab}{\sqrt{b^2-1}}$ 24. $\pm\frac{1}{2}\sqrt{3}$. 25. $\pm\frac{4}{5}a$.

Exercise 19. [Page 85.]

1. $\frac{1}{2}, 4$. 2. $\frac{5}{3}, -3$ 3. $\frac{3}{7}, -\frac{5}{2}$. 4. $-\frac{1}{3}, -\frac{1}{9}$.
5. $-2, \frac{8}{5}$. 6. $\frac{5}{7}, -\frac{2}{3}$. 7. $-(c+a), -(b-c)$.
8. $a, \dfrac{1}{a}$. 9. $a-2b+c, b-2a+c$.
10. $-\dfrac{a}{a+b}, \dfrac{b}{a+b}$. 11. $1, -\dfrac{a+b+c}{a+b}$. 12. $3a, 5a$.
13. $a(2+b), b(2-a)$. 14. $0, a+b$. 15. $\dfrac{a^2}{b}, \dfrac{b^2}{a}$.

Exercise 20. [Page 88.]

1. $-\frac{1}{2}, \frac{2}{3}$. 2. $\frac{5}{4}, -\frac{7}{6}$. 3. $\dfrac{5\pm\sqrt{-59}}{6}$. 4. $\frac{2}{5}, -\frac{1}{4}$.
5. $\dfrac{1\pm\sqrt{-2}}{3}$. 6. $\frac{4}{3}, \frac{8}{5}$. 7. 1, −1[illegible]/13. 8. $\dfrac{1\pm\sqrt{-3}}{2}$.
9. $1, \dfrac{a-b}{a+b}$. 10. $2a-b, b$. 11. $\dfrac{a}{a-b}, -\dfrac{b}{a+b}$.
12. a, b. 13. $1, \dfrac{b-c}{a-b}$. 14. $\dfrac{a}{b}, \dfrac{b}{a}$. 15. $\dfrac{c}{a+b}, \dfrac{c}{a-b}$.
16. $\dfrac{a+b}{a-b}, \dfrac{b-a}{b+a}$. 17. $1, \dfrac{c(a-b)}{a(b-c)}$. 18. $1, \dfrac{b+c-2a}{c+a-2b}$.

Exercise 21. [Page 89.]

1. $2, \frac{1}{3}$. 2. $\frac{3}{2}, \frac{4}{7}$. 3. $\frac{1}{3}, -\frac{1}{4}$. 4. $-a, \frac{3}{7}a$.
5. $\frac{1}{3}, -\frac{3}{2}$. 6. $\dfrac{1\pm 2\sqrt{2}}{3}$. 7. $\frac{3}{2}, \frac{1}{3}$. 8. $\dfrac{a}{2}, \dfrac{3}{b}$.
9. $1, \dfrac{a+b}{2-a}$. 10. $\dfrac{a}{b}, \dfrac{b}{c}$. 11. $-1, \dfrac{1-a}{1+a}$. 12. $\dfrac{1}{a}, \dfrac{1}{b}$.

Exercise 22. [Pages 91–92.]

1. $\pm 2,\ \pm\sqrt{10}$. **2.** $1,\ -2,\ \frac{1}{4}\{(\sqrt{5}-1)\pm\sqrt{-(10+2\sqrt{5})}\}$, $\frac{1}{4}\{-(\sqrt{5}+1)\pm\sqrt{-(10-2\sqrt{5})}\}$, $-\frac{1}{2}\{(\sqrt{5}-1)\pm\sqrt{-(10+2\sqrt{5})}\}$, $\frac{1}{2}\{(\sqrt{5}+1)\pm\sqrt{-(10-2\sqrt{5})}\}$. **3.** $16,\ \frac{50625}{2041}$. **4.** $\{a\pm\sqrt{a^2+b^2}\}^{\frac{1}{\ }}$

5. $\dfrac{-1\pm\sqrt{-3}}{2},\ 2\pm\sqrt{3}$. **6.** $(9\pm\sqrt{41})^{\frac{2}{3}}$. **7.** $\frac{1}{2},\ 2,\ \frac{1}{4}(3\pm\sqrt{-7})$.

8. $5,\ \dfrac{1\pm\sqrt{-23}}{2}$. **9.** $a,\ \dfrac{1\pm\sqrt{1+4a^3}}{2a^2}$.

10. $0,\ \frac{1}{2}\{-(a+b+c)\pm\sqrt{(a+b^2+c^2-2bc-2ca-2ab}\}$.

11. $-1,\ 1,\ 1,\ \dfrac{-1\pm 2\sqrt{-2}}{3}$. **12.** $1, -2,\ \frac{1}{2}\{-1\pm\sqrt{-15}\}$.

13. $a,\ b,\ -\frac{1}{2}(a+b)$. **14.** $1,\ -4,\ \frac{1}{2}(-3\pm\sqrt{-15})$.

15. $-1,\ -9,\ -5\pm\sqrt{-6}$. **16.** $2,\ -3,\ \frac{1}{8}(-3\pm\sqrt{-143})$.

17. $\frac{1}{2}\pm\frac{1}{4}\sqrt{\{\frac{1}{2}(1\pm\sqrt{33})\}}$. **18.** $1,\ 2,\ \frac{1}{2}\{3\pm\sqrt{-1}\}$.

19. ± 3; the roots $\pm\sqrt{14}$ being rejected.

20. $0,\ 3$; the roots $\frac{1}{2}(3\pm\sqrt{-23})$ being rejected.

21. $2,\ -5$; the roots $\frac{1}{2}(-3\pm\sqrt{241})$ being rejected.

22. $4,\ 9,\ \frac{1}{2}\{-33\pm\sqrt{-67}\}$. **23.** $a,\ b,\ \frac{1}{2}\{a+b\pm\frac{1}{3}(a-b)\sqrt{-63}\}$.

24. $4,\ -\frac{20}{9},\ \frac{1}{1632}\{-811\pm\sqrt{-13017839}\}$.

25. $\dfrac{(3\pm\sqrt{5})^5-2^5}{(3\pm\sqrt{5})^5+2^5}$. **26.** $\dfrac{(1\pm\sqrt{5})^m-2^m}{(1\pm\sqrt{5})^m+2^m}$.

27. $-1\pm 4\sqrt{2}$; the roots $4,\ -6$ being rejected.

28. $\pm m\left(\dfrac{n^5+1}{n^5-1}\right)$. **29.** $\left(\dfrac{p+q}{p-q}\right)^{12},\ \left(\dfrac{p-q}{p+q}\right)^{12}$.

30. $\pm\sqrt{(a^2+b^2+c^2)}$. **31.** $2\pm\sqrt{\{3\pm\frac{1}{3}\sqrt{57}\}}$.

32. $\dfrac{4ab\pm(a+b)\sqrt{6ab-a^2-b^2}}{(a-b)^2}$.

Exercise 23. [Page 95.]

4. (i) Identity. (ii) Equation. (iii) Equation.

Exercise 24. [Pages 105–107.]

1. (i) Real, irrational, unequal, (ii) Imaginary, unequal. (iii) Rational, unequal. (iv) Rational, unequal.

(v) Real, rational, equal. (vi) Real, rational, equal.
(vii) Rational, unequal. (viii) Rational, unequal.
(ix) Real, unequal. (x) Rational, unequal.

2. (i) $x^2-5x+6=0$. (ii) $x^2-3x-28=0$.
(iii) $15x^2-x-6=0$. (iv) $x^2-14x+1=0$.
(v) $x^2-2\sqrt{3}.x+2=0$. (vi) $9x^2-30x+7=0$.
(vii) $x^2-2ax+a^2-b=0$. (viii) $x^2-2ax+a^2-b^2=0$.
(ix) $(m^2-n^2)x^2+4mnx-(m^2-n^2)=0$.
(x) $(a+b)^2x^2+(a^2-b^2)x-ab=0$.

3. (i) $x^2-6x+2=0$. (ii) $x^2+2x-2=0$.
(iii) $x^2+6x+34=0$. (iv) $x^2-10x+27=0$.
(v) $x^2-2(a+b)x+2ab=0$. (vi) $x^2-10x+1=0$.
(vii) $x^2+x+1=0$. (viii) $x^2-2x-4=0$.

4. (i) ± 12. (ii) $\pm 8\sqrt{3}$. (iii) $-\frac{1}{2}$.

11. (i) $\frac{1}{2}$ or 2. (ii) $-\frac{2}{3}$. **23.** $bx^2-2ax+a=0$; $\dfrac{\sqrt{a}}{\sqrt{a}-\sqrt{(a-b)}}$.
24. $qx^2-(p^2-2q)+q=0$. **31.** 0.

Exercise 25. [Pages 111—114.]

1. (i) $\dfrac{\pm\sqrt{b^2-4ac}}{a}$. (ii) $\pm\dfrac{b\sqrt{b^2-4ac}}{a^2}$.

(iii) $\pm\dfrac{1}{a^3}(b^2-ac)\sqrt{b^2-4ac}$. (iv) $-\dfrac{bc}{a^2}$.

(v) $\dfrac{1}{a^4}(b^4-4ab^2c+2a^2c^2)$. (vi) $\dfrac{1}{a^4}(b^4-4ab^2c+3a^2c^2)$.

(vii) $\dfrac{b^2-2ac}{ac}$. (viii) $\dfrac{b^2-2ac}{c^2}$. (ix) $\pm\dfrac{(b^2-ac)\sqrt{b^2-4ac}}{a^2c}$.

2. (i) $2p^2+q$. (ii) $\dfrac{p^3-3pq}{q^3}$. (iii) $\dfrac{2}{p}$. (iv) $\dfrac{p^4-4p^2q+2q^2}{q^2}$.

3. $\dfrac{6abc-8b}{a^3}$; $\dfrac{40ab^3c-32b^5-10a^2bc^2}{a^5}$. **4.** $\dfrac{b^3-3abc}{a^3c^3}$.

6. $2(p^2-pp'+p'^2-2q-2q')$.

7. (i) $a^2x^2-(b^2-2ca)x+c^2=0$.
(ii) $a^3x^2+(b^3-3abc)x+c^3=0$.
(iii) $a^4x^2-2a^2(b^2-ac)x+b^2(b^2-2ac)=0$.

(iv) $c^2x^2-(b^2-2ac)x+a^2=0$.

(v) $a^2x^2+(m+1)abx+m(b^2-2ac)+(m^2+1)ac=0$.

(vi) $(a-b+c)x^2-2(a-c)x+a+b+c=0$.

8. (i) $x^2-(p+q)x+pq=0$. (ii) $x^2-px-2p^2+9q=0$.

(iii) $x^2-(p^2-2q+p)x+p^3-3pq+q^2+q=0$.

(iv) $qx^2-p(1+q)x+(1+q)^2=0$. (v) $p^2x^2-2px+1=0$.

(vi) $q^2(1+p+q)x^2-p(p^4+p^3q-4p^2q-3pq^2+2q^2)x+p^2q^2=0$.

9. $x^2-p^2x+p^2q=0$. **10.** $x^2-x+1=0$.

11. $4x^2-12x+5=0$. **12.** $x^2-50x+49=0$.

13. $\frac{1}{2}\{\gamma+\delta)\pm\sqrt{(\gamma+\delta)^2-4\alpha\beta}\}$. **14.** $3x^2-18x+2=0$.

15. $b^2=3ac$. **16.** $x^2+px+q=0$; since $\alpha=\dfrac{q}{\beta}$ and $\beta=\dfrac{q}{\alpha}$.

17. $x^2-px+q=0$; since $\alpha=\dfrac{q}{p-\alpha}$ and $\beta=\dfrac{q}{p-\beta}$.

19. $x^2-4mnx-(m^2-n^2)^2=0$. **20.** $x^2+4ax+2b=0$.

23. $q(m-n)^2+(mp-1)(np-1)=0$.

28. $x^2-pp'x+q(p'^2-2q')+q'(p^2-2q)=0$.

30. 3, 8. **31.** 5, 6. **32.** 1. **33.** $\frac{1421}{256}$.

Exercise 26 [Page 115.]

1. $a, 8-a$. **2.** $1, \dfrac{a-b}{b-c}$. **3.** $a, \dfrac{a(a-b)}{b-c}$. **4.** $a, \dfrac{1}{a}$.

5. $a, -\dfrac{1}{a}$. **6.** $a, -\dfrac{a}{a+1}$. **7.** $a, a-b-c$. **8.** $a, \dfrac{b-ac}{a+c}$.

9. $a+b, \dfrac{ab(a+b)}{a^2+b^2}$. **10.** $a+b+c, \frac{1}{2}(a+b)$. **11.** $c, -\dfrac{a^2+b^2}{(a+b)}$.

12. $1, \dfrac{(\sqrt{a}+\sqrt{b})^2+4}{(\sqrt{a}-\sqrt{b})^2-4}$. **14.** $1, \dfrac{c}{a}$. **15.** $-\dfrac{a^2+b^2+c(a+b)}{a+b+2c}$.

Exercise 27. [Pages 123—126.]

1. (i) 2, when $x=3$. (ii) $-\frac{37}{12}$, when $x=-\frac{11}{6}$.

(iii) $\frac{33}{8}$, when $x=\frac{5}{4}$. (iv) $-\frac{1}{16}$, when $x=\frac{7}{8}$.

(v) 13, when $x=1$. (vi) $-b^2$, when $x=0$.

2. (i) 8, when $x=\frac{3}{2}$. (ii) 7, when $x=-\frac{2}{5}$.
(iii) $\frac{81}{8}$, when $x=\frac{5}{4}$. (iv) $\frac{73}{8}$, when $x=\frac{5}{4}$.
(v) $\frac{121}{12}$, when $x=\frac{5}{6}$. (vi) a^2, when $x=0$.

6. When x lies between $\frac{1}{3}(4-\sqrt{5})$ and $\frac{1}{3}(4+\sqrt{5})$.

9. Positive when $x<\frac{3}{2}$, negative when $x>\frac{3}{2}$.

10. p lies between $+1$ and -1. 11. x lies between 3 and 5.

12. b^2-4cm^2 positive ; x lies between $\dfrac{-b\pm\sqrt{b^2-4cm^2}}{2m^2}$.

15. (i) 2 and 7. (ii) -1 and $\frac{7}{5}$. 18. 1.

21. a lies between $-\frac{2}{3}$ and $\frac{2}{3}$.

24. x lies between 5 and 6, and y between $-\frac{3}{2}$ and $-\frac{7}{2}$.

Exercise 28. [Pages 129—130.]

1. $\dfrac{a}{a'}=\dfrac{b}{b'}=\dfrac{c}{c'}$. 2. $(aa'-cc')^2=(ab'-bc')(a'b-b'c)$.

10. $2q+2q'-pp'$.

11. 7 or $32\frac{2}{3}$; $(2x-3y+5)(3x+y-4)$ or $(2x-3y+12)(3x+y-\frac{5}{3})$.

13. $2(ac-b)xy$. 14. a. 15. $-(\sqrt{p}\pm\sqrt{q})^2$.

16. $a'b^2+b'a^2=2aa'c$, $a'c^2=a^2c'$.

18. the roots of $(b^2-ac)z^2-(bc+a^2-a)z+c^2+ab-b=0$.

19. $\dfrac{bp-aq}{p}=\dfrac{cp-ar}{q}=\dfrac{pd}{r}$. 22. $a^2+b^2+c^2>2(bc+ca+ab)$.

Exercise 29. [Pages 137—138.]

1. $x=3, 5\frac{1}{8}$; $y=2, -\frac{1}{8}$. 2. $x=1, -\frac{77}{118}$; $y=2, -\frac{219}{118}$.

3. $x=2, -\frac{10}{93}$; $y=\frac{1}{5}, -\frac{307}{465}$. 4. $x=1, -\frac{20}{13}$; $y=2, \frac{37}{13}$.

5. $x=2, 1$; $y=1, 2$. 6. $x=7, 13$; $y=13, 7$.

7. $x=8, -5$; $y=5, -8$. 8. $x=1, 7$; $y=7, 1$.

9. $x=15, -13$; $y=13, -15$.

10. $x=7, y=2$. 11. $x=10, y=1$.

12. $x=1+\sqrt{-1}, 1-\sqrt{-1}$; $y=-1+\sqrt{-1}, -1-\sqrt{-1}$.

13. $x=3, 4\frac{4}{5}$; $y=8, 5$. 14. $x=5, 7$; $y=7, 5$.

15. $x=2, \frac{1}{10}$; $y=3, -\frac{1}{6}$. 16. $x=3\frac{1}{2}, 3\frac{2}{5}$; $y=2\frac{1}{3}, 2\frac{2}{5}$.

17. $x=3, 6$; $y=6, 3$. 18. $x=\frac{1}{5}, \frac{4}{5}$; $y=5, 20$.

19. $x = -1, 3$; $y = 1, -3$. **20.** $x = a$, $y = b$.

21. $x = \frac{1}{3}, -\frac{1}{2}$; $y = \frac{1}{2}, -\frac{1}{3}$. **22.** $x = \frac{ac}{a-b}$, $y = \frac{bc}{a-b}$.

23. $x = 0, 2$; $y = 0, 3$. **24.** $x = 3, 7$; $y = 7, 3$.

25. $x = 9, 6\frac{1}{2}$; $y = 4, 6\frac{1}{2}$. **26.** $x = 2$; $y = 3$.

27. $x = a + 2b, a - 2b$; $y = 2a + b, b - 2a$.

28. $x = 1, 9$; $y = 9, 1$.

29. $x = 3, -\frac{5}{2}$; $y = 2, -\frac{7}{2}$. **30.** $x = 2, 4$; $y = 4, 2$.

31. $x = \frac{1}{2}\left\{a \pm \sqrt{\frac{4b^3 - a^3}{3a}}\right\}$, $y = \frac{1}{2}\left\{a \mp \sqrt{\frac{4b^3 - a^3}{3a}}\right\}$.

32. $x = 7, -5$; $y = 5, -7$. **33.** $x = -3, 1$; $y = -2, 2$.

34. $x = 7, 4, -7, -4$; $y = 4, 7, -4, -7$.

35. $x = 3, 12$; $y = 12, 3$.

36. $x = \frac{a}{2b}\left\{-a \pm \sqrt{(a^2 + 4b^2)}\right\}$, $y = \frac{a}{2b}\left\{a \pm \sqrt{(a^2 + 4b^2)}\right\}$.

Exercise 30. [Pages 141 – 142.]

1. $x = \pm 2$, $y = \pm 1$. **2.** $x = \pm 1, \pm 2$; $y = \pm 3, \pm 1$.

3. $x = \pm 3, \pm\sqrt{2}$, $y = \pm 2, \mp 4\sqrt{2}$. **4.** $x = \pm 7$, $y = \pm 3$.

5. $x = \pm 3, \pm 2\sqrt{2}$; $y = \pm 1, \pm\sqrt{2}$.

6. $x = \pm\frac{5}{\sqrt{6}}$, $y = \pm\frac{1}{\sqrt{6}}$. **7.** $x = \pm 3, \pm\frac{5}{\sqrt{2}}$; $y = \pm 2, \pm\frac{1}{\sqrt{2}}$.

8. $x = \pm 2$, $y = \pm 3$; $x = \pm 3\sqrt{\frac{6}{11}}$, $y = \pm 2\sqrt{\frac{6}{11}}$.

9. $x = \pm 4, \pm\frac{38}{3}\sqrt{\frac{3}{29}}$; $y = \pm 1, \pm\sqrt{\frac{2}{29}}$.

10. $x = \pm 3, \pm\frac{5}{3}$; $y = \pm 5, \pm\frac{13}{3}$.

11. $x = \pm 2$, $y = \pm 3$. **12.** $x = \pm 3, \pm 13\sqrt{\frac{1}{91}}$; $y = \pm 2, \pm 5\sqrt{\frac{1}{91}}$.

13. $x = \pm 1, \pm\frac{13}{21}\sqrt{21}$; $y = \pm 3, \pm\frac{2}{21}\sqrt{21}$.

14. $x = \pm\sqrt{\frac{a^2 + b^2}{2}}$; $y = \pm\frac{a^2 - b^2}{\sqrt{2(a^2 + b^2)}}$.

15. $x = \pm 2, \pm\sqrt{\frac{2}{5}}$; $y = \pm\frac{1}{2}, \mp 2\sqrt{\frac{2}{5}}$.

16. $x = \frac{2}{3}\sqrt{-3}, -\frac{2}{3}\sqrt{-3}$; $y = -\frac{2}{3}\sqrt{-3} \pm \sqrt{7}, \frac{2}{3}\sqrt{-3} \pm \sqrt{7}$.

17. $x = \pm 2, \pm 5\sqrt{\frac{5}{67}}$; $y = \pm 3, \mp\frac{34}{5}\sqrt{\frac{5}{67}}$.

18. $x = \pm 1, \pm\frac{7}{3}$; $y = \mp 2, \pm\frac{2}{3}$.

19. $x = \pm 1, \pm\frac{1}{2}\sqrt{-11}$; $y = \pm 1, \mp\sqrt{-11}$.

20. $x = \pm\sqrt{a+b}, \pm\sqrt{a-b}\ ;\ y = \pm\sqrt{a+b}, \mp\sqrt{a-b}.$

21. $x = \pm\frac{a}{\sqrt{a+b}}\ ;\ y = \frac{\quad}{\sqrt{a+b}}.$

22. $x = \pm\frac{p}{\sqrt{ap+bq}}\ ;\ y = \pm\frac{q}{\sqrt{ap+bq}}.$

23. $x = \frac{1}{\sqrt{7}}, -\frac{1}{\sqrt{7}}\ ;\ y = \pm\sqrt{\frac{32}{7}}, \pm\sqrt{\frac{32}{7}}.$

24. $x = \pm 3,\ \pm 8\sqrt{\frac{37}{127}}\ ;\ y = \pm 4,\ \pm 6\sqrt{\frac{37}{127}}.$

25. $x = \pm\sqrt{\left\{\frac{a}{2b}(ab \pm \sqrt{a^2b^2 - 4c^4})\right\}},$

$y = \pm\sqrt{\left\{\frac{b}{2a}(ab \mp \sqrt{a^2b^2 - 4c^4})\right\}}.$

Exercise 31. [Page 144.]

1. $x=2, 3\ ;\ y=3, 2.$
2. $x=8, -3\ ;\ y=3, -8.$
3. $x=3, 1, 2\pm 5\sqrt{-1}\ ;\ y=1, 3, 2\mp 5\sqrt{-1}.$
4. $x=2, 6, 4\pm 10\sqrt{-1}\ ;\ y=6, 2, 4\mp 10\sqrt{-1}.$
5. $x=7, -3, 2\pm 7\sqrt{-1}\ ;\ y=3, -7, -2\pm 7\sqrt{-1}.$
6. $x=6, 4, 5\pm\sqrt{-151}\ ;\ y=-4, -6, -5\pm\sqrt{-151}.$
7. $x=4, 2, 3\pm\sqrt{-19}\ ;\ y=2, 4, 3\mp\sqrt{-19}.$
8. $x=2, -1, \frac{1}{2}\pm\sqrt{-11}\ ;\ y=1, -2, -\frac{1}{2}\pm\sqrt{-11}.$
9. $x=3, 6\ ;\ y=6, 3.$
10. $x=\frac{2\pm\sqrt{2}}{4}, \frac{1\pm\sqrt{2}}{2};\quad y=\frac{2\mp\sqrt{2}}{4}, \frac{1\mp\sqrt{2}}{2}.$
11. $x=3, 1\ ;\ y=1, 3.$
12. $x=a^2, ab\ ;\ y=b^2, ab.$
13. $x=a, y=b.$
14. $x=a, b\ ;\ y=b, a.$
15. $x=\frac{ac}{a+b},\ y=\frac{bc}{a+b}.$
16. $x=2, 1\ ;\ y=1, 2.$

Exercise 32. [Pages 146—147.]

1. $x=3, 6\ ;\ y=6, 3.$ 2. $x=4, 5, -5\pm\sqrt{5}\ ;\ y=5, 4, -5\mp\sqrt{5}.$
3. $x=7, -\frac{39}{5}\ ;\ y=11, -\frac{56}{5}.$
4. $x=1, -1\ ;\ y=\frac{1}{2}, -2.$
5. $x=\pm\frac{\sqrt{a^2+1}}{a};\ y=\pm\sqrt{a^2+1}.$

6. $x=\frac{a}{2}(1\pm\sqrt{3}), \frac{a}{2}\left(1\pm\frac{1}{\sqrt{3}}\right);\ y=\frac{a}{2}(1\mp\sqrt{3}), \frac{a}{2}\left(1\mp\frac{1}{\sqrt{3}}\right).$

7. $x=0, 5, -\frac{1}{2}(1\pm\sqrt{-11})$; $y=0, 5, -\frac{1}{2}(1\mp\sqrt{-11})$.

8. $x=2, 4$; $y=4, 2$.

9. $x=b, b, a-b, a-b, \frac{1}{2}(a\pm\sqrt{5a^2-4ab})$;
$y=a, b-a, a, b-a, \frac{1}{2}(b\pm\sqrt{5b^2-4ab})$.

10. $x=3, 5$; $y=5, 3$.

11. $x=\pm\frac{1}{\sqrt{3}}(2b-a), \pm a\sqrt{-1}$; $y=\pm\frac{1}{\sqrt{3}}(2a-b), \mp b\sqrt{-1}$.

12. $x=3, 3, -\frac{1}{3}, -\frac{1}{3}$; $y=3, -\frac{1}{3}, 3, -\frac{1}{3}$.

13. $x=3, 1, -1, -3$; $y=1, 3, -3, -1$.

14. $x=\pm 5, \pm 3$; $y=\pm 3, \pm 5$.

15. $x=\pm\left(\frac{1}{b}+\frac{1}{a}\right), \pm\left(\frac{1}{b}-\frac{1}{a}\right);\ y=\pm\left(\frac{1}{b}-\frac{1}{a}\right), \pm\left(\frac{1}{b}+\frac{1}{a}\right).$

16. $x=b, \frac{a^2}{b}$; $y=a, \frac{b^2}{a}$. **17.** $x=0, 12$; $y=0, 4$.

18. $x=9, y=4$. **19.** $x=2, 1$; $y=1, 2$.

20. $x=\frac{7}{5}, -\frac{1}{5}$; $y=-\frac{1}{5}, \frac{7}{5}$.

21. $x=a, \frac{3ab-a^2}{a+b}$; $y=b, \frac{3ab-b^2}{a+b}$.

22. $x=\left(\frac{a}{b}\right)^{\frac{b}{a-b}}, y=\left(\frac{a}{b}\right)^{\frac{a}{a-b}}$.

23. $x=\frac{1}{2}\{4m+1\mp\sqrt{(8m+1)}\}, y=\frac{1}{2}\{-1\pm\sqrt{(8m+1)}\}$.

24 $x=2, y=2$. **25.** $x=-1, 3$; $y=1, -3$.

26. $x=3, 1$; $y=1, 3$. **27.** $x=\frac{a}{2}(\sqrt{5}\pm 1)$; $y=\frac{b}{2}(\sqrt{5}\mp 1)$.

28. $x=3, 2, -3\pm\sqrt{3}$; $y=2, 3, -3\mp\sqrt{3}$.

29. $x=9, 1$; $y=1, 9$.

30. $x=4, -2, 1\pm\sqrt{\frac{3}{11}}$; $y=2, -4, -1\pm\sqrt{\frac{3}{11}}$.

31. $x=\pm 6, \mp 6$; $y=\pm 3, \pm 3$. **32.** $x=1, \frac{5}{3}$; $y=2, \frac{2}{3}$.

33. $x=\pm\frac{1}{\sqrt{2}}, \frac{\pm\sqrt{2}\pm\sqrt{6}}{4}$; $y=\frac{\pm\sqrt{2}\pm\sqrt{6}}{4}, \pm\frac{1}{\sqrt{2}}$.

34. $x=2, 1, -2, -1, 2\sqrt{-1}, \sqrt{-1}, -\sqrt{-1}, -2\sqrt{-1}$;
$y=1, 2, -1, -2, \sqrt{-1}, 2\sqrt{-1}, -2\sqrt{-1}, -\sqrt{-1}$.

35. $x=0, \frac{3}{2}, 1\pm\sqrt{-3}$; $y=0, \frac{1}{2}, 1\mp\sqrt{-3}$.

Exercise 33. [Page 151.]

1. $x=4, 0$; $y=5, 33$; $z=6, 40$.
2. $x=2, \frac{484}{357}$, $y=3, \frac{1439}{357}$, $z=1, -\frac{57}{357}$.
3. $x=0, 2a$; $y=b, -b$; $z=c, -c$.
4. $x=\pm 1$; $y=\pm 3$; $z=\pm 5$.
5. $x=\pm 10$; $y=\mp 2$; $z=\pm 14$.
6. $x=3$; $y=4$; $z=5$. 7. $x=\pm 1$; $y=\pm 3$; $z=\pm 5$.
8. $x=\pm 1$; $y=\pm 2$; $z=\pm 3$.
9. $x=\pm \dfrac{l}{\sqrt{al+bm+cn}}$; $y=\pm \dfrac{m}{\sqrt{al+bm+cn}}$;

$$z=\pm \frac{n}{\sqrt{al+bm+cn}}.$$

10. $x=ka(b^2-c^2)$, $y=kb(c^2-a^2)$, $z=kc(a^2-b^2)$, where $k=\infty$.

Exercise 34. [Pages 156–157.]

1. $x=\pm \dfrac{bc}{a}$; $y=\pm \dfrac{ca}{b}$; $z=\pm \dfrac{ab}{c}$.
2. $x=0, \pm \sqrt{\left\{\dfrac{2(b+c-a)}{(c+a-b)(a+b-c)}\right\}}$;

$$y=0, \pm \sqrt{\left\{\frac{2(c+a-b)}{(a+b-c)(b+c-a)}\right\}};$$

$$z=0, \pm \sqrt{\left\{\frac{2(a+b-c)}{(b+c-a)(c+a-b)}\right\}}.$$

3. $x=\pm \sqrt[4]{\dfrac{a^3}{bc}}$; $y=\pm \sqrt[4]{\dfrac{b^3}{ca}}$; $z=\pm \sqrt[4]{\dfrac{c^3}{ab}}$.
4. $x=\pm 2$; $y=\pm 5$; $z=\pm 6$. 5. $x=\pm 1$; $y=\pm 2$; $z=\pm 3$.
6. $x=\pm a$; $y=\pm b$; $z=\pm c$.
7. $x=\pm \frac{97}{19}$, $y=\pm \frac{121}{19}$, $z=\pm \frac{143}{19}$.
8. $x=\pm a^2.\sqrt{\left\{\dfrac{b^2+c^2-a^2}{(c^2+a^2-b^2)(a^2+b^2-c^2)}\right\}}$: and similar values for y and z.
9. $x=5, -7$; $y=3, -5$; $z=6, -8$.
10. $x=\pm \sqrt{\frac{2}{3}}$; $y=\pm \sqrt{\frac{1}{6}}$; $z=\pm \sqrt{\frac{3}{2}}$.
11. $x=4, \frac{60}{7}$; $y=6, \frac{66}{7}$; $z=2, -6$.

12. $x = -m \pm \sqrt{\left\{\dfrac{(b+m^2)(c+m^2)}{a+m^2}\right\}}$, and similar values for y and z.

13. $x = \frac{12}{5}$; $y = \frac{12}{7}$; $z = -12$.

14. $x = \frac{1}{2}\{-a \pm \sqrt{a^2+4p^2}\}, \dfrac{p^2-a^2}{a}, a, a$;

$y = \frac{1}{2}\{-a \pm \sqrt{a^2+4p^2}\}, a, \dfrac{p^2-a^2}{a}, a$;

$z = \frac{1}{2}\{-a \pm \sqrt{a^2+4p^2}\}, a, a, \dfrac{p^2-a^2}{a}$.

15. $x = 0,\ 4a,\ \frac{3}{2}a,\ -a,\ -a$; $y = 0,\ 4a,\ -a,\ \frac{3}{2}a,\ -a$; $z = 0,\ 4a,\ -a,\ -a,\ \frac{3}{2}a$.

16. $x = \pm\frac{1}{2}\left(\dfrac{ca}{b} + \dfrac{ab}{c} - \dfrac{bc}{a}\right)$,
$y = \pm\frac{1}{2}\left(\dfrac{ab}{c} + \dfrac{bc}{a} - \dfrac{ca}{b}\right)$,
$z = \pm\frac{1}{2}\left(\dfrac{bc}{a} + \dfrac{ca}{b} - \dfrac{ab}{c}\right)$.

17. $x = \pm\dfrac{b^2+c^2}{2\sqrt{(a^2+b^2+c^2)}}$;
$y = \pm\dfrac{c^2+a^2}{2\sqrt{(a^2+b^2+c^2)}}$;
$z = \pm\dfrac{a^2+b^2}{2\sqrt{(a^2+b^2+c^2)}}$.

18. $x = \pm\dfrac{a(b^2+c^2)}{2bc}$; $y = \pm\dfrac{b(c^2+a^2)}{2ca}$; $z = \pm\dfrac{c(a^2+b^2)}{2ab}$.

19. $x = \pm 2$; $y = 0$; $z = \mp 1$.

20. $x = \pm 6$; $y = \pm 4$; $z = \pm 5$.

21. $x = \pm\dfrac{b-c}{a}$; $y = \pm\dfrac{c-a}{b}$; $z = \mp\dfrac{a-b}{c}$.

22. $x = 1, \frac{3}{10}$; $y = 2, \frac{5}{6}$; $z = 3, \frac{7}{3}$.

23. $x = 0, \frac{1}{2}a$; $y = 0, \frac{1}{2}b$; $z = 0, \frac{1}{2}c$.

24. $x = 8, 2$; $y = 4, 4$; $z = 2, 8$.

25. x, y, z are the different arrangements of $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$.

26. x, y, z are the different arrangements of 2, 3, 4.

27. x, y, z are the different arrangements of 3, 5, 7.

28. x, y, z are the different arrangements of a, 0, 0.

29. x, y, z are the different arrangements of $1, -1, 3$.

30. $x = 0,\ \frac{1}{2}(1 \pm \sqrt{-1}),\ \frac{1}{2}(-1 \pm \sqrt{6})$; $y = \frac{1}{2}(1 \pm \sqrt{-1}),\ 0,\ \frac{1}{2}(-1 \mp \sqrt{6})$; $z = \frac{1}{2}(1 \mp \sqrt{-1}),\ \frac{1}{2}(1 \mp \sqrt{-1}),\ 2$.

31. $y = \pm 3$, $y = \mp 3$, $z = \pm 1$.

32. x, y, z are the different arrangements of 1, 2, 3,

33. $x=a-b$; $y=b-c$; $z=c-a$.
34. x, y, z are the different arrangements of 3, 3, 4.
35. x, y, z are the different arrangements of 1, 2, 3.
36. x, y, z are the different arrangements of 2, 3, 4.

Exercise 36. [Pages 174—175.]

25. $x^2+y^2-6x-8y=0$. 26. 1.
29. 1·4, 1·7, 2·2, 2·6. 30. 6·7.

Exercise 37. [Pages 179—180.]

1. (i) 1, −2. (ii) 3, −2. (iii) 3, ·5. (iv) 0, 1·5.
(v) −1·5, −1·5. (vi) ·3, −·5. (vii) ·7, −3. (viii) 3·6, −·6.
(ix) 2·7, −·2. (x) 2·4, −·9.
2. (i) $x=4$, $y=-3$; or, $x=-3$, $y=4$.
(ii) $x=12$, $y=-5$; or, $x=-5$, $y=12$.
(iii) $x=3·9$, $y=-·6$; or, $x=-2·1$, $y=3·4$.
(iv) $x=1$, $y=3$; or, $x=-·3$, $y=3$.
(v) $x=0$, $y=0$; or, $x=1·25$, $y=2·5$.
(vi) $x=3$, $y=2$; or, $x=2$, $y=3$. (vii) $x=5$, $y=3$.
(viii) $x=2$, $y=1$; or, $x=-2$, $y=-1$; or, $x=1·5$, $y=1·3$; or, $x=-1·5$, $y=-1·3$. (ix) $x=3$, $y=2·5$; or, $x=2·2$, $y=2·7$.
(x) $x=2$, $y=3$; or, $x=-2$, $y=-3$.
3. 2, 3. 4. 3, −1. 5. 1·6, −·6. 6. (3·6, 4·8).
7. −5·5. 8. 17. 9. (1, 2), (−·5, ·5). 10. 7·5.
11. ·4, or −2·4. 12. 2, 2 ; −1, −1·5 ; imaginary roots.

Exercise 38. [Page 184.]

1. (i) $21\sqrt{-1}$. (ii) $3\sqrt{-1}$. (iii) $-30\sqrt{15}$. (iv) $-48\sqrt{-6}$.
(v) 6. (vi) $-\frac{1}{9}\sqrt{3}$. (vii) −54. (viii) −373248.
2. (i) i. (ii) −1. (iii) $-i$. (iv) 1.
(v) $-i$. (vi) −1. (vii) i. (viii) 1.

Exercise 39. [Pages 191—192.]

1. (i) $2+3i$. (ii) $\frac{1}{2}(1+i)$. (iii) $1-i$. (iv) $\frac{4abi}{a^2+b^2}$. (v) $\frac{8ab(a^2-b^2)i}{(a^2+b^2)^2}$.
2. (i) $\sqrt{2}$. (ii) 5. (iii) $\frac{13}{25}$. (iv) $\frac{1}{2}$. (v) $(x^2+y^2)^{\frac{n}{2}}$.

3. (i) $9(4+\sqrt{2}i)$. (ii) $30+16\sqrt{15}$. (iii) $6+\sqrt{10}$.

4. (i) $5+2\sqrt{6}$. (ii) $\frac{1}{2}(1+i)$. 5. $5(-1+8i)$.

6. $2abc-\{a^3+b^3+c^3+(b+c)(c+a)(a+b)\}i$.

7. (i) $\pm(2+i)$. (ii) $\pm\frac{1}{\sqrt{2}}(5-3i)$. (iii) $\pm(\sqrt{2}+3i)$. (iv) $\pm(1+\sqrt{2}i)$.

(v) $\pm\{\sqrt{(a+bi)}+\sqrt{(a-bi)}\}$. (vi) $\pm\frac{1}{\sqrt{2}}\left\{\sqrt{(x^2+1)}+i\sqrt{(x^2-1)}\right\}$.

(vii) $\pm\{x-i\sqrt{(x^2-1)}\}$. (viii) $\pm\frac{1}{\sqrt{2}}\left\{\sqrt{(x^2+x+1)}+i\sqrt{(x^2-x+1)}\right\}$.

8. $\frac{x}{y}+\frac{1}{2}i-\frac{y}{x}$. 9. $m+ni$. 10. $6i$. 17. 12.

18. $24\sqrt{-1}-17$. 19. $\frac{2}{9}\{2\sqrt{-3}-3\sqrt{-2}+2\sqrt{6}+3\}$.

Exercise 40. [Page 196.]

2. $(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$. 9. -238.

Exercise 41. [Page 199.]

1. (i) 81. (ii) -104. (iii) $-\cdot19$. (iv) $a+(n-2)b$

(v) n^2+n-1. 2. -3. 3. 20th term. 4. 1.

5. 13. 6. 95. 7. 34. 8. $p+q-r$, 0, 1.

9. 1, 4, 7,... 10. $\frac{r(m-n)-(mq-np)}{p-q}$.

Exercise 42. [Page 201.]

1. (i) 11. (ii) $\frac{1}{2}$. (iii) $\frac{3}{2}(a+b)$. (iv) $2ab$. (v) $ax+by$.

2. (i) 11, 17, 23, 29. (ii) 6, 11, 16, 21, 26, 31, 36.

(iii) 7, 12, 17, 22, 27, 32, 37, 42, 47, 52.

(iv) $\frac{1}{4}(b+3a)$, $\frac{1}{2}(b+a)$, $\frac{1}{4}(3b+a)$.

(v) $4a+3b$, $7a+7b$, $10a+11b$. 3. $n=11$.

Exercise 43. [Pages 202—203.]

1. 5050. 2. 2550. 3. 900. 4. 12. 5. 248.

6. 2480. 7. -21. 8. -42. 9. 520.

10. $21(\sqrt{2}-20)$. 11. $\frac{1}{2}n(n-3+2\sqrt{2})$. 12. $\frac{1}{2}n(n+1)$.

13. $\frac{1}{2}n\{(n+1)a-(n+3)b\}$. 14. $n(x^2+y^2)-n(n-3)xy$.

15. $\frac{1}{2}n(n+1)-\frac{1}{2}m(m-1)$. **16.** 246950. **17.** 2754000.
18. 300. **19.** $\frac{1}{2}n(3n+1)$. **20.** 3775.

Exercise 44. [Page 206.]

1. $\frac{1}{2}n(6n^2-3n-1)$. **2.** $\frac{2}{3}n(n+1)(2n+1)$.
3. $\frac{1}{2}n(6n^2+21n+23)$. **4.** $\frac{1}{3}n(4n^2+12n+11)$.
5. $\frac{1}{3}n(16n^2+6n+23)$. **6.** $\frac{1}{4}n(27n^3+18n^2-9n-4)$.
7. $n^2(2n^2-1)$. **8.** $\frac{1}{2}n(6n^2-57n+179)$.
9. $\frac{1}{12}n(n+1)(n+2)(3n+5)$. **10.** $\frac{1}{3}n(4n^2+6n-1)$.
11. $\frac{1}{6}n(n+1)(n+2)$. **12.** $\frac{1}{3}n(n+1)(n+2)$.
13. $n(n+1)(2n^2+6n+1)-3n$.
14. $\frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$.
15. $\frac{1}{6}n(2n^2+3n+19)$. **16.** $\frac{1}{3}n(n^2+3n+5)$.
17. $\frac{1}{12}n^2(n+1)^2(2n^2+2n-1)$.
18. $\frac{1}{42}n(n+1)(2n+1)(3n^4+6n^3-3n+1)$.
19. $\dfrac{n}{n+1}$. **20.** $\dfrac{n}{2(3n+2)}$

Exercise 45. [Pages 208–209.]

1. (i) 4096. (ii) $537\frac{3}{5}$. (iii) $-\frac{1}{256}$. (iv) $-486\sqrt{3}$. (v) $512\sqrt{6}$.
2. $\dfrac{\sqrt{3}}{3^{n-1}}$. **3.** $\frac{11}{4}\left(\frac{2}{3}\right)^{n-1}$. **4.** $\frac{27}{16}$, $\frac{2}{3}$, $\frac{1}{3}$.
5. 8th term. **6.** $\frac{1}{32}$. **7.** 6th term.
8. $\left(\dfrac{a^{n-q}}{b^{n-p}}\right)^{\frac{1}{p-q}}$. **9.** pth term $=\sqrt{mn}$; qth term $=m\left(\dfrac{n}{m}\right)^{\frac{p}{2q}}$.
10. 4, $-\frac{16}{3}$, $\frac{64}{9}$, $-\frac{256}{27}$······ **13.** $\frac{1}{3}$, 1, 3, 9,··· or, $-\frac{2}{3}$, 2, −6, 18,···
14. 4, −8, 16, −32, 64,······

Exercise 46. [Page 211.]

1. (i) $\pm 5\sqrt{3}$. (ii) ± 6. (iii) ± 6. (iv) $\pm x\sqrt{ab}$.
(v) $\pm(x-y)\sqrt{xy}$. (vi) $\pm(2x-5)(3x+2)$.
2. (i) ± 12, 36, ± 108. (ii) $1\frac{2}{3}$, $1\frac{1}{9}$, $\frac{20}{27}$, $\frac{40}{81}$.
(iii) $\frac{1}{6}$, $-\frac{1}{3}$, $\frac{2}{3}$, $-\frac{4}{3}$. (iv) $\pm\frac{8}{5}$, $\frac{4}{5}$, $\pm\frac{2}{5}$, $\frac{1}{5}$, $\pm\frac{1}{10}$.

(v) $\frac{32}{27}, \frac{16}{9}, \frac{8}{3}, 4, 6, 9.$ (vi) $1\frac{1}{5}, \frac{4}{5}, \frac{8}{15}, \frac{16}{45}.$
(vii) $\pm 4\sqrt{2}, 4, \pm 2\sqrt{2}, 2, \pm\sqrt{2}, 1, \pm\frac{1}{2}\sqrt{2}.$
(viii) $\pm 225, 2025, \pm 18225.$ **5.** 9, 12. **8.** 27, 3.

Exercise 47. [Pages 212—213.]

1. $-170.$ **2.** 265720. **3.** $2\frac{20}{81}.$
4. $\frac{1365}{2048}.$ **5.** $-341.$ **6.** $\frac{1281}{512}.$
7. $73\frac{227}{256}.$ **8.** $-\frac{31}{2}(2-\sqrt{2}).$ **9.** $364\sqrt{2}(\sqrt{3}+1).$
10. $\frac{9841}{3}(3+\sqrt{3}).$ **11.** $\frac{4}{3}\{1-(-\frac{1}{2})^n\}.$
12. $2^{\frac{3-n}{2}}(\sqrt{2}+1)(2^{\frac{n}{2}}-1).$ **13.** $\frac{1}{2}\cdot\frac{3^{n-1}}{3^n-1}.$ **14.** $\frac{1}{3}\{1-(\frac{1}{10})^n\}.$
15. $\frac{1}{2}(5+3\sqrt{3})\{1-(2-\sqrt{3})^n\}.$ **16.** $\frac{2}{3}(\sqrt{6}-2)\{1-(-1)^n(\frac{3}{2})^{\frac{n}{2}}\}.$
17. $\frac{(a-b)^2}{2b(a+b)}\left\{\left(\frac{a+b}{a-b}\right)^n-1\right\}.$ **18.** $\frac{a+b}{(a-b)^{n-2}}\left\{\frac{(a-b)^n-1}{(a-b)-1}.\right\}$
19. $a^p.\frac{a^{nq}-1}{a^q-1}.$ **20.** $a.\frac{1-(-r)^n}{1+r}.$
21. $\frac{1}{2}(4+3\sqrt{2})\{1-(\sqrt{2}-1)^n\}.$ **22.** $\left(1-\frac{1}{2^n}\right)-\frac{1}{2}\left(1-\frac{1}{3^n}\right).$
23. (i) $3(3^n-1).$ (ii) $\frac{a^4}{a^4+1}\left\{(-1)^n a^{4n}-1\right\}.$
24. $\frac{ac^{\frac{1}{n+1}}-ca^{\frac{1}{n+1}}}{a^{\frac{1}{n+1}}-c^{\frac{1}{n+1}}}.$ **25.** $\frac{1-r^n}{ar^{n-1}(1-r)^n}.$

Exercise 48. [Pages 215—216.]

1. 2. **2.** $1\frac{1}{2}.$ **3.** 6. **4.** $1\frac{4}{5}.$ **5.** 1.
6. $5\frac{2}{5}.$ **7.** 1. **8.** $\frac{13}{24}.$ **9.** $2\frac{1}{4}.$ **10.** $2\sqrt{2}.$
11. 1728. **12.** $27\frac{3}{7}.$ **13.** $\frac{1}{2}(4+3\sqrt{2}).$
14. $\frac{1}{2}(5+3\sqrt{3}).$ **15.** $\frac{1}{1-2x}.$ **16.** $\frac{1}{1+x}.$
17. (i) $\frac{116}{495}$; (ii) $\frac{313}{900}$; (iii) $\frac{3}{11}$; (iv) $\frac{203}{990}.$

Exercise 49. [Pages 219—220.]

1. $\frac{4}{9}.$ **2.** The 4 terms before 4 are $2\frac{2}{9}, 2\frac{1}{2}, 2\frac{6}{7}, 3\frac{1}{3}$; and the 4 terms after $6\frac{2}{3}$ are 10, 20, ∞, -20.

3. $\frac{60}{16-n}$. **4.** $\frac{3}{7}$. **6.** $\frac{ac(n-1)}{a(m-1)-c(m-n)}$.

9. (i) $5\frac{1}{11}$. (ii) $12\frac{3}{14}$. (iii) $\frac{x^2-y^2}{x}$.

10. (i) $1\frac{29}{121}$, $1\frac{29}{48}$, $2\frac{8}{11}$, $4\frac{7}{17}$.

(ii) $\frac{2(a^2-b^2)}{3a-b}$, $\frac{a^2-b^2}{a}$, $\frac{2a^2-b^2}{2a+b}$.

13. $a:b:c=1:1:1$, or $1:-2:4$.

Exercise 50. [Pages 233–235.]

1. (i) 95040. (ii) 3628800. (iii) $15 . 14 . 13 \cdots (16-r)$.

3. $n=10$. **4.** $n=7$. **5.** $n=14$. **6.** $r=4$.

10. 120. **11.** 1320. **12.** 30. **13.** 720 ; 120.

14. 719 ; 119. **15.** 120. **16.** 120. **17.** 300.

18. 357840. **19.** 720. **20.** 2880. **21.** 576.

22. $n(n-1)(n-2)(n-3)\ldots\ldots(n-m+1)$.

23. 6720. **24.** (i) 209 ; (ii) 5 ; (iii) 143.

25. 3600. **26.** 100800. **27.** (i) 2016 ; (ii) 3024.

28. 201600 odd, 161280 even. **29.** 154.

Exercise 51. [Pages 244—246.]

1. 27405 ; 658008. **2.** 20. **3.** $n=10$; $r=3$.

4. $r=n-1$. **8.** 120. **9.** 210 ; 84.

10. 10800 ; 360. **11.** 126. **12.** 28 ; 84.

13. 36. **14.** 63. **15.** 196.

16. 68. **17.** 120 ; 35. **18.** (i) 40. (ii) 116.

19. $\frac{1}{8}n(n-1)(n-2)(n-3)$. **20.** $\frac{\underline{|m}\;\underline{|m+1}}{\underline{|m-n+1}}$. **21.** 3.

22. 200. **23.** $\frac{\underline{|890}}{\underline{|70}\;\underline{|820}}$. **24.** ·25.

25. 246. **26.** 990. **27.** 56, 21.

28. 10 ; 92378.

Exercise 52. [Pages 251–253.]

1. $\underline{|9}$. 2. $\underline{|10}$, $\underline{|9}$ or $\frac{1}{2}\underline{|9}$. 3. $\frac{1}{2}\underline{|6}$.

4. $\underline{|7} \times \underline{|7}$ or $\frac{1}{2}\underline{|7} \times \underline{|7}$ 5. $\dfrac{\underline{|m}\,\underline{|m-1}}{\underline{|m-n}}$ or $\dfrac{\underline{|m}\,\underline{|m-1}}{2\underline{|m-n}}$.

6. 840. 7. 403920. 8. 816000.

9. 126. 10. 344. 11. 34650. 12. 5775.

13. $\dfrac{\underline{|mn}}{\{\underline{|m}\}^n}$. 14. $\dfrac{mn}{n\{\underline{|m}\}^n}$. 15. 2^n.

16. $\dfrac{\underline{|m+n}}{mn}$. 17. 31. 18. 255.

19. 63. 20. $24 \times \dfrac{\underline{|36}}{\{\underline{|9}\}^4}$; $576 \times \dfrac{\underline{|36}}{\{\underline{|9}\}^4}$.

Exercise 53. [Pages 255–257.]

1. (i) 10080. (ii) 151200. (iii) 10810800.

2. 3632428800. 119750400. 3. 50. 4. 120.

5. 7560, 60. 6. 1260. 7. 1023. 8. 1260 ; 900 ; 660.

9. (i) 10080 ; (ii) 120 ; (iii) 360 ; (iv) 84. 10. 21599.

11. $\dfrac{[illegible]}{\underline{|5}(\underline{|4})^2(\underline{|6})^3}$. 12. 840. 13. 9000.

14. 7775. 15. 254. 16. 78125.

17. 125. 18. 243. 19. 480.

20. 28672, 12287. 21. m^n.

Exercise 54. [Page 258.]

1. 34. 2. 1023. 3. 6719.

4. 18. 5. 260865. 6. 3255.

Exercise 55. [Pages 260–261.]

1. 160 ; 2724. 2. 167. 3. (i) 399 ; (ii) 2454.

4. 286. 5. 3630. 6. 159. 7. (i) 113 ; (ii) 2190.

8. 60. 9. 1638 ; 39.

Exercise 56. [Pages 263 – 265.]

1. 3999960. **2.** 2599980. **3.** 22222200. **4.** 359391.

5. (i) $(\underline{|13})^4 \times \underline{|4}$. (ii) $2^4 \times \underline{|4}$. **6.** 19. **7.** 250047.

8. 9. **10.** 144000. **11.** $\dfrac{\underline{|2n-p-q}}{\underline{|n-p}\,\underline{|n-q}} \times (\underline{|n})^2$.

12. $\dfrac{\underline{|20}}{\underline{|10}\,\underline{|10}}$. **13.** 66, 132. **14.** 6. **15.** 15840.

16. 8085. **17.** $\dfrac{n(n-1)(n-2)}{\underline{|3}} - \dfrac{p(p-1)(p-2)}{\underline{|3}} + 1$.

18. 15. **20.** $\frac{1}{2}mn(m-1)(n-1)$.

Exercise 57. [Pages 273 – 274.]

1. $1+6x+15x^2+20x^3+15x^4+6x^5+x^6$.

2. $a^7+7a^6b+21a^5b^2+35a^4b^3+35a^3b^4+21a^2b^5+7ab^6+b^7$.

3. $x^5+10x^4+40x^3+80x^2+80x+32$.

4. $32x^5-80x^4+80x^3-40x^2+10x-1$.

5. $16x^4+96x^3+216x^2+216x+81$.

6. $a^{12}-6a^{10}bc+15a^8b^2c^2-20a^6b^3c^3+15a^4b^4c^4$
$-6a^2b^5c^5+b^6c^6$.

7. $x^8-16x^7y+112x^6y^2-448x^5y^3+1120x^4y^4-1792x^3y^5$
$+1792x^2y^6-1024xy^7+256y^8$.

8. $1+\frac{7}{2}x+\frac{21}{4}x^2+\frac{35}{8}x^3+\frac{35}{16}x^4+\frac{21}{32}x^5+\frac{7}{64}x^6+\frac{1}{128}x^7$.

9. $\frac{243}{32}x^5-\frac{135}{8}x^4y+15x^3y^2-\frac{20}{3}x^2y^3+\frac{20}{27}xy^4-\frac{32}{243}y^5$.

10. $3360x^4$. **11.** $\frac{56}{2304}a^4b^8$. **12.** $-\dfrac{120x^4}{a^4}$.

13. $\frac{5103}{16}x^4y^5$. **14.** $760xy^6$. **15.** $\dfrac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}x^n$.

16. 352. **17.** $140\sqrt{2}$. **18.** $2+8x^2-8x^4$.

19. $2(64x^3+2160x^2+4860x+729)$. **20.** $1120x^4y^4$.

21 $20x^3$. **22.** $-\dfrac{35x}{y}$, $\dfrac{35y}{x}$.

23. $\dfrac{\underline{|2n+1}}{\underline{|n}\,\underline{|n+1}}a^{n+1}x^n$, $\dfrac{\underline{|2n+1}}{\underline{|n+1}\,\underline{|n}}a^nx^{n+1}$

24. $\left(\frac{a}{b}\right)^{2n+1} + (2n+1)\left(\frac{a}{b}\right)^{2n-1} + {}^{2n+1}C_2\left(\frac{a}{b}\right)^{2n-3}$
$+ \left(\frac{b}{a}\right)^{2n+1}$; ${}^{2n+1}C_r\left(\frac{a}{b}\right)^{2n-2r+1}$; ${}^{2n+1}C_n\,\frac{a}{b}$; ${}^{2n+1}C_{n+1}\,\frac{b}{a}$

25. $312y^2$. 26. -252. 27. $210c^{12}$.

28. $\frac{\underline{|n}}{\underline{|m}\,\underline{|n-m}}y^n$ 29. $(-1)^{n-r}\frac{\underline{|2n+1}}{\underline{|n-r}\,\underline{|n+r+1}}$.

30. $-\frac{\underline{|35}}{\underline{|15}\,\underline{|20}}a^{20}b^{15}$ 31. $\frac{1792}{9}$. 32. 495. 33. $\frac{\underline{|2n}}{\underline{|n}\,\underline{|n}}$. 34. n.

Exercise 58. [Page 278.]

1. 9th and 10th terms $= {}^{14}C_8 2^6.3^8$.
2. 5th and 6th terms $= {}^{12}C_4.(\frac{5}{8})^4$. 3. 3rd term $= \frac{15}{4}$.
4. 4th and 5th terms $= {}^{13}C_3.5^{10}.2^3$.
5. 4th and 5th terms $= \frac{224}{9}$.
6. 7th term $= 5580130\frac{1}{2}$. 7. $\frac{2816}{27}$. 8. $\frac{\underline{|14}}{\underline{|6}\,\underline{|8}}.3^8.2^6$.

9. If $n=1$, the 2nd and 3rd terms; if $n=2$, the 2nd term; for all other values of n, the 1st term. 10. $\frac{15}{16}$ and $\frac{16}{15}$.

Exercise 60. [Pages 283 – 284.]

1. 1. 5. 12. 9. (i) 1·01206; (ii) ·99700: (iii) 1·00350. 13. 0. 14. When n is even, ${}^nC_{\frac{n}{2}}$ and when n is odd ${}^nC_{\frac{n-1}{2}}\left(x+\frac{1}{x}\right)$.

Exercise 61. [Pages 292—294.]

1. $1+\frac{1}{2}x+\frac{3}{8}x^2+\frac{5}{16}x^3+\cdots\cdots$; $\frac{1.3.5\cdots\cdots(2r-1)}{2^r\underline{|r}}x^r$.

2. $1+\frac{5}{2}x+\frac{15}{8}x^2+\frac{5}{16}x^3-\cdots\cdots$; $(-1)^{r-3}\frac{5.3\,1.1.3\cdots(2r-7)}{2^r\underline{|r}}x^r$.

3. $1-\frac{2}{3}x+\frac{5}{9}x^2-\frac{40}{81}x^3+\cdots\cdots$; $(-1)^r\frac{3.2.5.8\cdots\cdots(3r-1)}{3^r\underline{|r}}x^r$.

4. $1-\frac{1}{3}x-\frac{1}{9}x^2-\frac{5}{81}x^3-\cdot$ $\frac{1.2.5.8\cdots\cdots(3r-4)}{3^r\underline{|r}}x^r$.

5. $1+2x+5x^2+\frac{40}{3}x^3+\cdot$ $\frac{2.5.8\cdots(3r-1)}{\underline{|r}}x^r$.

6. $\sqrt[3]{3}.\,(1+\frac{2}{9}x-\frac{4}{81}x^2+\frac{40}{2187}x^3-\ldots)$;

$$(-1)^{r-1}.\frac{1.2.5.8.\cdots(3r-4)}{3^{2r}\lfloor r}.2^r x^r.$$

7. $\frac{1}{a}-\frac{b}{a^2}x+\frac{b^2}{a^3}x^2-\frac{b^3}{a^4}x^3+\cdots$; $(-1)^r\frac{b^r}{a^{r+1}}x^r$.

8. $a^{-\frac{3}{2}}\left(1+\frac{3}{4}.\frac{x^2}{a^2}+\frac{3.7.x^4}{4.8\,a^4}+\frac{3.7.11x^6}{4.8.12.a^6}+\cdots\cdots\right)$;

$$\frac{3.7.11\cdots(4r-1)}{4.8.12\cdots4r}.a^{-\frac{3}{2}-2r}x^{2r}.$$

9. $a-3a^{\frac{4}{3}}x^{-\frac{1}{3}}+6a^{\frac{5}{3}}x^{-\frac{2}{3}}-10a^2x^{-1}+\cdots\cdots$;

$$(-1)^r.\frac{(r+1)(r+2)}{1.2}.a^{\frac{r+3}{3}}x^{-\frac{r}{3}}.$$

10. $1+3x+\frac{3.5}{1.2}x^2+\frac{3.5.7}{1.2.3}x^3+\cdots\cdots$; $\frac{3.5.7\cdots\cdots(2r+1)}{\lfloor r}x^r$.

11. $1+x+\frac{1.5}{1.2}x^2+\frac{1.5.9}{1.2.3}x^3+\cdots\cdots$; $\frac{1.5.9\cdots(4r-3)}{\lfloor r}x^r$.

12. $1+\frac{x}{n}+\frac{(n+1)}{n^2\lfloor 2}x^2+\frac{(n+1)(2n+1)}{n^3\lfloor 3}x^3+\cdots\cdots$;

$$\frac{(n+1)(2n+1)(3n+1)\cdots\cdots\{(r-1)n+1\}}{n^r\lfloor r}x^r.$$

13. $a\left(1+\frac{1}{2}\frac{x}{a^2}-\frac{1}{8}\frac{x^2}{a^4}+\frac{1}{16}\frac{x^3}{a^6}-\frac{5}{128}\frac{x^4}{a^8}+\cdots\right)$.

14. $3.\frac{1.3.5\cdots(2r-7)}{2^{r-1}\lfloor r-1}x^{r-1}$.

15. $\frac{(n+1)(2n+1)(3n+1)\cdots\{(r-2)n+1\}}{n^{r-1}\lfloor r-1}a^{-\frac{n(r-1)+1}{n}}x^{r-1}$.

17. $-\frac{5}{16}$. **19.** 5th term $=-\frac{7}{243}x^4$.

24. $3\{1+\frac{1}{3}.\frac{x^2}{a^3}+\frac{2}{9}.\frac{x^4}{a^6}+\frac{14}{81}.\frac{x^6}{a^9}+\frac{35}{243}.\frac{x^8}{a^{12}}+\cdots\}$;

$$\frac{1.4.7\cdots\cdots(3r-2)}{3^{r-1}\lfloor r}.\frac{x^{2r}}{a^{3r}}.$$

25. $\frac{(r+1)(r+2)(r+3)}{1.2.3}$. **28.** $x=\frac{1}{2}y-\frac{1.3}{2.4}y^2+\frac{1.3.5}{2.4.6}y^3-\cdots\cdots$

30. 1, 0.

Exercise 62. [Page 297.]

1. 2nd term $= 1\frac{1}{4}$. 2. 3rd term $= \frac{11}{2}$.
3. 3rd term $= \frac{45}{14}$. 4. 1st term $= 1$.
5. 3rd term $= \frac{1612}{225}$. 6. 1st term $=$ 2nd term $= (\sqrt{2})^{\sqrt{2}}$.
7. 1st term $=$ 2nd term $= x^x$. 8. 1st term $= 1$.
9. 12th term $=$ 13th term $= \dfrac{12.13.14.15}{1.2.3.4} \cdot \left(\dfrac{3}{4}\right)^{11}$.
10. 1st term $= \dfrac{1}{\sqrt{2}}$.

Exercise 63. [Pages 298—299.]

1. 4·89898. 2. 9·94987. 3. 1·99776.
4. 125·1500. 5. 1·01059. 6. 1·4953.

Exercise 64. [Page 301]

1. $\sqrt[3]{4}$. 2. $\sqrt[3]{\frac{9}{4}}$. 3. $\sqrt{3}-1$. 4. $\frac{2}{5}\sqrt{5}$. 5. 2.
6. $2\frac{1}{4}$. 7. $\frac{4}{3}(2\sqrt{3}-3)$. 8. $\frac{1}{2}$. 14. $\frac{1}{3}n(n+1)(n+2)$.
15. $\dfrac{\lfloor 2n-1}{\lfloor n-1 \, \lfloor n}$.

Exercise 65. [Pages 304–305.]

1. 121. 2. $4n$. 3. $r(r^2+3)$
4. $2^{n-2}(9n^2+15n+8)$. 6. $1-3x+13x^2$. 11. $\dfrac{a^r-b^r}{a-b}.x^r$.

Exercise 66. [Pages 310–311.]

1. (i) 4. (ii) $\frac{4}{3}$. (iii) $\frac{5}{3}$. (iv) -6. (v) $\frac{4}{3}$. (vi) 0.
2. (i) 0. (ii) $2\log 2+\log 5$. (iii) $\log_2 3$. 5. $\frac{1}{3}k$.

Exercise 67. [Page 316.]

1. 4, 4, 2, -4.
2. 2·89432, ·49831, $\bar{1}$·72016, $\bar{3}$·92428, $\bar{2}$·40140.
3. 2·14601··· 4. 31 digits; 30 zeros. 9. $-\frac{1}{8}$.
10. 14 digits. 11. ·30103, ·47712. 12. ·69897, ·84510.
13. 7th. 16. $m = \dfrac{n}{n-1}$.

Exercise 68. [Page 317.]

1. $\cdot 631$.   2. $\cdot 1760213$.   3. $-\cdot 991$.

4. $5\cdot 30$.   5. $2\cdot 079$.   6. $5\cdot 90$.

7. $\left(\dfrac{\log b}{\log a}\right)^2$.   8. $\dfrac{4\log b-3\log a}{2\log a-3\log b}$.

9. $\dfrac{\log 3}{\log a+\log b}$, $\dfrac{\log 3}{\log a-\log b}$.

10. $\dfrac{\log(\log c)-\log(\log a)}{\log b}$.   11. $x=\cdot 903$, $y=\cdot 854$.

12. $x=\dfrac{\log 2.\log b}{(\log b)^2-(\log a)^2}$, $y=\dfrac{\log 2.\log a}{(\log a)^2-(\log b)^2}$.

13. $x=2\cdot 71$, $y=1\cdot 71$.   14. $x=\cdot 03$.

15. $x=1$, $y=0$.   16. $x=y=z=\log_{abc}m$.

17. $\log x=\frac{1}{5}(a+3b)$, $\log y=\frac{1}{5}(a-2b)$.

Exercise 69. [Pages 324–325.]

1. $\frac{1}{2}(e+e^{-1})$.   2. $\frac{1}{2}(e-e^{-1})$.   3. $e+e^{-1}-2$.

4. $3e$.   5. $5e$.   6. $3e$.

7. $\frac{11}{2}e-2$.   8. $5e$.   9. $e^{a^2}-e^{b^2}$.

10. $\dfrac{e-e^a}{1-a}$.   21. (i) $\cdot 61$; (ii) $\cdot 8187$.

22. $$\left(1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\cdots\right)+\left(1+\frac{2}{\lfloor 2}+\frac{3}{\lfloor 3}+\cdots\right)x$$
$$+\left(1+\frac{2^2}{\lfloor 2}+\frac{3^2}{\lfloor 3}+\cdots\right)\frac{x^2}{\lfloor 2}+\left(1+\frac{2^3}{\lfloor 2}+\frac{3^3}{\lfloor 3}+\cdots\right)\frac{x^3}{\lfloor 3}$$
$$+\left(1+\frac{2^4}{\lfloor 2}+\frac{3^4}{\lfloor 3}+\cdots\right)\frac{x^4}{\lfloor 4}+\cdots$$

23. $\dfrac{(-1)^n}{\lfloor n}\{a-bn+cn(n-1)\}$.   27. $e^{\frac{x}{a}}$.

Exercise 70. [Pages 331–332.]

5. (i) $\dfrac{x}{1-x}-\log_e(1-x)$; (ii) $\dfrac{1}{1-x}+\dfrac{1}{x}\log_e(1-x)$.

6. $-\{(2+3)x+\frac{1}{2}(2^2+3^2)x^2+\frac{1}{3}(2^3+3^3)x^3+\ldots\}$; $-\dfrac{1}{n}(2^n+3^n)$.

7. $x+\frac{1}{2}x^2-\frac{2}{3}x^3+\frac{1}{4}x^4+\frac{1}{5}x^5-\frac{2}{6}x^6+\cdots$; the co-eff. of x^n is $\frac{1}{n}$ or $-\frac{2}{n}$ according as n is not or is a multiple of 3.

8. $x+\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{3}{4}x^4+\frac{1}{5}x^5+\cdots\cdots\cdots$; the co-eff. of x^{2n} is $-\frac{3}{2n}$ or $\frac{1}{2n}$, according as n is even or odd, and the co-eff. of x^{2n+1} is $2n+1$.

15. (i) $1-\log_e 2$; (ii) $\frac{3}{4}-\log_e 2$;
(iii) $\frac{2}{3}\log_e 2-\frac{5}{12}$; (iv) $\frac{1}{6}\log_e 2-\frac{1}{24}$.

19. $2\left\{\frac{x^2}{2}-\left(1+\frac{1}{2}\right)\frac{x^3}{3}+\left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{x^4}{4}-\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\frac{x^5}{5}+\cdots\right\}$

Exercise 71. [Pages 341 – 343.]

1. Rs. 250. 10 as. 8p. ; Rs. 1504.
2. $4\frac{1}{2}$ p. c.
3. Rs. 578. 5 as.
4. $9\frac{1}{4}\frac{3}{8}$ p.c. ; Rs. 2956. 4 as.
5. 11·463 p. c.
7. Rs. 41. 15 as. 11·29344p. ; Rs. 201. 15 as. 11·29344p.
8. 14·2 years nearly.
9. £ 785. 2s. 2d. nearly.
11. Rs. 3850.
12. Rs. 6815. 12 as. 9·6p.
13. £ 1979. 2s. 6d. nearly.
14. Rs. 31200.
15. Rs. 140927. 9 as. 7·2p. nearly.
16. Rs. 5359. 5 as. 4p.
17. £ 703. 8s. 9d.
18. Rs. 2500.
19. 25 years' purchase.
20. (i) Rs. 20000 ; (ii) Rs. 17276. 12 as. nearly.
21. £ 622. 9s. 7d.
22. Rs. 290.
23. £ 1738 nearly.
24. 20 years' purchase.

CALCUTTA UNIVERSITY

INTERMEDIATE EXAMINATION PAPERS

1928

1. *Either*, If α, β be the roots of $x^2-px+q=0$, find the equation whose roots are $2\alpha-\beta$ and $2\beta-\alpha$.

If α, β and γ, δ be the roots of $x^2-px+q=0$ and $x^2-p'x+q'=0$ respectively, find the value of

$$(\alpha-\gamma)(\beta-\delta)+(\beta-\gamma)(\alpha-\delta).$$

Or, Solve $ax^2+by^2=a+b$, $x+y=1$.

If α, β and γ, δ be the two solutions of the above equations, find the value of $(\alpha-\gamma)(\beta-\delta)+(\beta-\gamma)(\alpha-\delta)$.

2. *Either*, Show that the sum of the combinations of n gs, taken 1, 2,......n at a time, is 2^n-1.

how that the number of all possible selections of one or ore questions from eight given questions, each question having an alternative, is 3^8-1.

Or, Find the number of combinations of n different things taken r at a time.

Find the number of different straight lines obtainable by joining n different points on a plane, no three of which are collinear, with the exception of p points which are collinear.

3. *Either*, Prove the Binomial Theorem for a negative integral index, assuming it to be true for a positive integral index.

Verify by actual multiplication up to four terms that the product of

$$1+mx+\frac{m(m-1)}{1.2}x^2+\frac{m(m-1)(m-2)}{1.2.3}x^3+\ldots\ldots$$

and $$1+nx+\frac{n(n-1)}{1.2}x^2+\frac{n(n-1)(n-2)}{1.2.3}x^3+\ldots$$

is $$1+(m+n)x+\frac{(m+n)(m+n-1)}{1.2}x^2$$
$$+\frac{(m+n)(m+n-1)(m+n-2)}{1.2.3}x^3+\ldots\ldots$$

Or, Prove that

$$e^x=1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\cdots+\frac{x^n}{\lfloor n}+\ldots\ldots$$

Verify, by actual multiplication up to four terms, that the product of

$$1+x+\frac{x^2}{1.2}+\frac{x^3}{1.2.3}+\ldots$$

and
$$1+y+\frac{y^2}{1.2}+\frac{y^3}{1.2.3}+\ldots\ldots$$

is
$$1+(x+y)+\frac{(x+y)^2}{1.2}+\frac{(x+y)^3}{1.2.3}+\ldots\ldots$$

1929

1. *Either*, If α and β are the roots of $x^2+px+q=0$, find the values of, in terms of p and q,

(i) $(1+\alpha+\alpha^2)(1+\beta+\beta^2)$;

(ii) $(\omega\alpha+\omega^2\beta)(\omega^2\alpha+\omega\beta)$, where ω is an imaginary cube root of unity.

Or, (i) Solve $x^2+y^2=a$, $x+2y=1$.

(ii) If $x+2y=1$, find the least value of x^2+y^2.

2. *Either*, Find directly the number of permutations of n different things taken three at a time.

Find how many different words can be formed with five given letters, of which three are consonants and two are vowels, no two consonants being juxtaposed in any word.

Or, If x varies as y when z is constant, and x varies as z when y is constant, show that x varies as yz when both y and z are variable.

The mass m of a body varies as density p when the volume v is constant and varies as volume v when the density p is constant. If unit mass be defined as mass of a body of unit volume and unit density, show that $m=pv$.

3. *Either*, Write down the expansion of $(1+x)^n$ in ascending powers of x, when n is a positive integer.

If the above expansion the successive co-efficients are $a_0, a_1, a_2, \ldots\ldots a_n$, show that

$$a_0+2a_1+3a_2+\cdots\cdots+(n+1)a_n=2^n+n.2^{n-1}.$$

Or, Write down the expansion of e^x in ascending powers x.

Show that

$$e^2-e=1+\frac{1+2}{\lfloor 2}+\frac{1+2+2^2}{\lfloor 3}+\ldots\ldots$$

1930

1. *Either*, (i) If $x=\alpha$, $y=\beta$ is a solution of $x+y=a$, $xy=b$, show that α, β are roots of the equation $x^2-ax+b=0$.

(ii) Solve $(x+a)(y+b)=c^2$, $x+y=a+b$.

Or, (i) Find the condition that the roots of $ax^2+bx+c=0$ may be imaginary.

(ii) Show that $x^2+y^2+z^2-yz-zx-xy$ is the product of two imaginary factors.

2. *Either*, (i) Find the number of combinations of n different things taken r at a time.

(ii) If ${}^{2n}C_r={}^{2n}C_{r+2}$, find r.

Or, (i) Prove the Binomial Theorem for a positive integral exponent.

(ii) If in the expansion of $(1+x)^{2n+1}$ the co-efficients of x^r and x^{r+1} be equal, find r.

1931

1. *Either*, Solve $x+y=a+b$, $\frac{a}{x}+\frac{b}{y}=2$.

If α and β be the roots of $x^2+px+q=0$, show that $\frac{\alpha}{\beta}$ is a root of the equation $qx^2-(p^2-2q)x+q=0$.

Or, If x varies as y when z is constant, and x varies as z when y is constant, show that x varies as yz when both y and z are variable.

Two globes of gold have their radii equal to r and r'; they are melted and formed into a single globe. Find its radius. (The volume of a globe varies as the cube of the radius.)

2. Find the number of permutations of n things, taken all at a time, when p of them are exactly alike of one kind, q of them exactly alike of a different kind, and the rest are all different.

Out of 9 Swarajists and 6 Ministerialists, how many different committees can be formed, each consisting of 6 Swarajists and 3 Ministerialists ?

3. *Either*, Obtain the term free from x in the expansion of $\left(x+\frac{1}{x}\right)^{\cdot}$

If $y=2x+3x^2+4x^3+\cdots\cdots$ express x in a series of ascending powers of y, up to the third term.

Or, Write down the expansion of e^x and $\log_e(1+x)$, in ascending powers of x.

If $y=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots$ show that $x=y+\frac{y^2}{\lfloor 2}+\frac{y^3}{\lfloor 3}+\cdots\cdots$

1932

1. *Either*, Express the roots of the equation $q^2x^2-(p^2-2q)x+1=0$ in terms of those of $x^2+px+q=0$.

Solve $\frac{a}{x}+\frac{b}{y}=2a$, $(a-b)x+(a+b)y=a+b$.

Or, If x varies as y when z is constant, and varies as z when y is constant, show that x varies as yz when both y and z vary.

A playground whose length and width are in the ratio of 8 : 7 has two-thirds of it reserved for accommodation. If the width is to be diminished by one-ninth, in what ratio should the length be increased in order that the accommodation may be trebled ?

2. *Either*, Find ${}^nC_r$ by any method.

A candidate is required to answer six out of ten-questions which are divided into two groups each containing five questions, and he is not permitted to attempt more than four from each group. In how many different ways can he make up his choice ?

Or, Prove the Binomial Theorem when the exponent is a positive integer.

Expand $\left(\frac{a}{b}+\frac{b}{a}\right)^{2n+1}$, giving in particular, the general term and the two middle terms.

3. Prove the following :

(i) $\log(ab)=\log a+\log b$;

(ii) $\log_a m=\log_b m \cdot \log_a b$;

(iii) $\log m=2\left\{\frac{m-1}{m+1}+\frac{1}{3}\left(\frac{m-1}{m+1}\right)^3+\frac{1}{5}\left(\frac{m-1}{m+1}\right)^5+\cdots\cdots\right\}$.

1933

1. Prove that a quadratic equation cannot have more than two roots. Find the condition that the roots may be real. Is it possible for a quadratic equation to have one real and one imaginary root ?

If α, β be the roots of the equation $ax^2+bx+c=0$, find the equation whose roots are α^2 and β^2.

2. (i) Find the sum to n terms of the series

$$1^3+2^3+3^3+\ldots\ldots$$

(ii) Show that a real value of x will satisfy the equation

$$\frac{1-ix}{1+ix}=a-ib,$$

if $a^2+b^2=1$, a and b being real and i denoting $\sqrt{-1}$.

3. Find the number of combinations of n different things taken r at a time. How would the result be affected if two of the things were exactly similar ?

In a municipal corporation there are 20 councillors and 8 aldermen. How many committees can be formed consisting of 5 councillors and 3 aldermen ?

4. Prove the Binomial Theorem for a positive integral index.

If $(1+x)^n=c_0+c_1x+c_2x^2+\cdots\cdots+c_nx^n$, prove that

$$c_0{}^2+c_1{}^2+c_2{}^2+\cdots+c_n{}^2=\frac{(2n)\,!}{n\,!\;n\,!}.$$

5. (i) Expand $\log(1+x)$ in a power-series in x. Is the expansion valid for all values of x ?

(ii) If $y=x+x^2+2x^3+\cdots\cdots+\frac{(2n)\,!}{(n+1)\,!}x^{n+1}\cdot$ *ad inf.*, prove that $y^2-y+x=0$.

1934

1. (i) If r be the ratio of the roots of the equation $ax^2+bx+c=0$, show that

$$\frac{(r+1)^2}{r}=\frac{b^2}{ac}.$$

(ii) Determine the values of m for which

$$3x^2+4mx+2=0 \text{ and } 2x^2+3x-2=0$$

may have a common root.

2. If x varies as y when z is constant, and x varies as z when y is constant, show that x varies as yz when y and z vary.

Apply the principles of variation to find how long 25 men will take to plough 30 acres, if 5 men take 9 days to plough 10 acres of land.

3. Find the number of permutations of n things when p of them are a's, q b's, r c's, and the rest different.

A library has 5 copies of one book, 4 copies of each of two books, 6 copies of each of three books, and single copies of eight books. In how many ways can all the books be arranged ?

4. (i) Find the general term in the expansion of $(1+x)^n$, and obtain the middle term.

(ii) Find the term independent of x in the expansion of $\left(x^2+\frac{1}{x}\right)^{12}$.

5. Prove the Exponential Theorem.

Show that $$\frac{e-1}{e+1}=\frac{\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}+\ldots\ldots}{\frac{1}{1!}+\frac{1}{3!}+\frac{1}{5!}+\ldots\ldots}$$

6. (i) Prove that $\log_a m=\log_b m\times\log_a b$; and hence or otherwise deduce that $\log_b a\times\log_c b\times\log_a c=1$.

(ii) Given $\log 104=2{\cdot}017033$, find in how many years Rs.100 will become Rs.1000 at 4 per cent. compound interest. Give the answer correct to two places of decimals.

1935

1. (i) If α and β be the roots of the equation $ax^2+bx+c=0$, express $\alpha^2+\beta^2$ in terms of a, b, c.

(ii) Prove that for real values of x the expression $3x^2-6x+8$ can never be less than 5.

2. (i) If $x=\dfrac{\sqrt{a+2b}+\sqrt{a-2b}}{\sqrt{a+2b}-\sqrt{a-2b}}$, show that $bx^2-ax+b=0$.

(ii) Solve the equations : $x^y=y^x$, and $x=2y$.

3. (i) Prove that, with the usual notation, ${}^nC_r=\frac{n}{r}.{}^{n-1}C_{r-1}$.

(ii) At an election there are five candidates and three members to be elected, and a voter is entitled to vote for any number of candidates not greater than the number to be elected. In how many ways may a voter choose to vote ?

4. (i) Establish the Binomial Theorem for a positive integral index. Deduce that the sum of the binomial co-efficients in the expansion of $(1+x)^n$ is 2^n.

(ii) In the expansion of $(1+x)^{m+n}$, where m and n are positive integers, prove that the co-efficients of x^m and x^n are equal.

5. (i) Expand $\log_e (1+x)$ in ascending powers of x. For what values of x is the expansion valid ?

(ii) What sum should be paid for an annuity of £100 a year to be paid for 40 years, money being supposed to be worth 4 per cent. per annum ? (Logarithmic tables may be consulted.)

6. (i) Prove the Exponential Theorem.

(ii) Show that

$$1+\frac{1+2}{\underline{|2}}+\frac{1+2+3}{\underline{|3}}+\frac{1+2+3+4}{\underline{|4}}+\ldots\ldots=\frac{3e}{2}.$$

1936

1. (i) For what values of m will the equation

$$x^2-2(5+2m)x+3(7+10m)=0$$

have (1) equal roots, (2) reciprocal roots ?

(ii) If α and β are the roots of the equation $ax^2+bx+c=0$, find the equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

2. (i) If $x+y \quad x-y$ show that

(1) $x^2+y^2 \quad xy$, and (2) $ax+by \quad px+qy$,

a, b, p, q being all constants.

(ii) Solve $\frac{1}{x}+\frac{1}{y}=\frac{1}{2}$, $x+y=9$.

3. (i) Find the number of permutations of n different things taken r at a time.

(ii) In how many of the permutations of 10 things taken 4 at a time, will one particular thing (1) always occur, (2) never occur ?

4. (i) Prove the Binomial Theorem for a positive integral index.

(ii) Find the term independent of x in the expansion of $\left(2x+\frac{1}{3x^2}\right)^9$.

5. (i) Find $\frac{1}{\sqrt[5]{e}}$ correct to four places of decimals.

(ii) Show that $\frac{2}{\lfloor 1}+\frac{4}{\lfloor 3}+\frac{6}{\lfloor 5}+\frac{8}{\lfloor 7}+\cdots\cdots$ to inf. $=e$.

6. (i) Show that $7\log\frac{16}{15}+5\log\frac{25}{24}+3\log\frac{81}{80}=\log 2$.

(ii) Find the present value of an annuity of Rs. 300 per annum for 5 years at 4 per cent. (Logarithmic tables may be consulted.)

1937

1. (i) If one root of the equation $x^2-px+q=0$ be twice the other, show that $2p^2=9q$.

(ii) If x be real, show that the least value of $4x^2-4x+1$ is 0, and the corresponding value of x is $\frac{1}{2}$.

2. Solve the equations :

(i) $2\sqrt{(x+5)}-\sqrt{(2x+8)}=2$;

(ii) $x+y=\frac{5}{6}$, $\frac{1}{x}-\frac{1}{y}=1$.

3. (i) Find the number of combinations of n different things taken r at a time.

(ii) From six gentlemen and four ladies a committee of five is to be formed. In how many ways can this be done so as to include at least one lady ?

4. (i) Prove that in the expansion of $(a+b)^n$, the co-efficients of terms equidistant from the beginning and end are equal.

(ii) Write down the co-efficient of x^{10} in $\frac{1+x}{(1-x)^3}$.

5. (i) If $a^{3-x}b^{5x}=a^{x+5}b^{3x}$, show that $x\log\left(\frac{b}{a}\right)=\log a$.

(ii) A man invests £ 30 a year in a Savings Bank which pays $2\frac{1}{2}$ per cent. per annum on all deposits. What will be the total amount at the end of 20 years ?

6. (i) Express $\frac{1}{2}(e^{ix}+e^{-ix})$ in ascending powers of x, where $i=\sqrt{(-1)}$.

(ii) Show that $e^{-1}=\frac{2}{3!}+\frac{4}{5!}+\frac{6}{7!}+\cdots\cdots$ to infinity.

1938

1. (i) If α and β are the roots of $x^2+px+q=0$, form the equation whose roots are $(\alpha-\beta)^2$ and $(\alpha+\beta)^2$.

(ii) If x is real, prove that $\frac{x^2+34x-71}{x^2+2x-7}$ can have no value between 5 and 9.

2. Solve the equations :

(i) $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{5}{2}$, $x+y=10$.

(ii) $x(y+z)=5$, $y(x+z)=8$, $z(x+y)=9$.

3. (i) Find the number of permutations of n things taken all together when the things are not all different.

(ii) A cricket team consisting of eleven players is to be selected from two groups consisting of six and eight players respectively. In how many ways can the selection be made on the supposition that the group of six shall contribute no fewer than four players ?

4. (i) Write down the co-efficients of the $(r+1)$th term in the expansion of $(1-2x)^{-\frac{1}{2}}$.

(ii) If $c_0, c_1, c_2, \cdots\cdots c_n$ be the co-efficients in the expansion of $(1+x)^n$, prove that

$$c_1+2c_2+3c_3+\cdots\cdots+nc_n=n.2^{n-1}.$$

5. (i) Expand $e^{\frac{x}{e}}$ in ascending powers of x up to x^4.

(ii) Express in terms of e

$$\left(1+\frac{1}{1.2}+\frac{1}{1.2.3}+\ldots\right)\left(1-\frac{1}{1.2}+\frac{1}{1.2.3}-\ldots\right).$$

6. (i) Find to two places of decimals the value of x from the equation $6^{3-4x}.\,4^{x+5}=8$.

(ii) In how many years will a sum of money double itself at 5 per cent. compound interest ?
[Given log 2=0·3010300, log 3=0·4771213, log 7=0·8450980.]

1939

1. (i) If α, β are the roots of the equation $ax^2+bx+c=0$, form an equation whose roots are $\alpha+\frac{1}{\beta}$ and $\beta+\frac{1}{\alpha}$.

(ii) If $x^2+px+q=0$ and $x^2+qx+p=0$ have a common root, show that either $p=q$ or $1+p+q=0$.

2. Solve the equations :

(i) $xy+x+y=27$, $\frac{1}{x}+\frac{1}{y}=\frac{1}{2}$.

(ii) $xz+y=7z$, $yz+x=8z$, $x+y+z=12$.

3 (i) Find the number of combinations of n different things taken r at a time.

(ii) How many words of 2 vowels and 3 consonants can be formed from an alphabet of 5 vowels and 17 consonants, the letters of the word being all different ?

4. (i) Prove the Binomial Theorem for a positive integral index.

(ii) Find the co-efficient of x^n in the expansion of $\frac{(1+x)^2}{(1-x)^2}$.

5. (i) Prove the exponential theorem.

(ii) Find the sum to infinity of $1+\frac{2^3}{\lfloor 2}+\frac{3^3}{\lfloor 3}+\frac{4^3}{\lfloor 4}+\cdots\cdots$

6. (i) Prove that

$$x^{\log y-\log z}\times y^{\log z-\log x}\times z^{\log x-\log y}=1.$$

(ii) Find the present value of an annuity of Rs. 2400 for 20 years at $4\frac{1}{2}$ per cent. compound interest per annum. [Given log 1·045 = ·0191 and log 4·150 = ·6180.]

1940

1. (i) If one root of the equation $ax^2+bx+c=0$ be four times the other, show that $4b^2=25ac$.

(ii) Prove that the expression $\frac{x^2-3x+4}{x^2+3x+4}$ lies between $\frac{1}{7}$ and 7 for all real values of x.

2. Solve the equations :

(i) $x+\frac{4}{y}=1$, $y+\frac{4}{x}=25$.

(ii) $x^2+y^2+z^2=84$, $x+y+z=14$, $xz=y^2$.

3. (i) Find the number of permutations of n different things taken r at a time.

(ii) Find how many words can be formed of the letters in the word *failure*, the four vowels always coming together.

4. (i) Find the middle term or terms in the expansion of $(x+a)^n$.

(ii) In the expansion of $(1+x)^n$, prove that the sum of the co-efficients of the odd terms is equal to the sum of the co-efficients of the even terms.

5. (i) Expand $\log_e (1+x)$ in a series of ascending powers of x, when $x<1$.

(ii) Sum to infinity $\frac{1}{2.3}+\frac{1}{4.5}+\frac{1}{6.7}+\ldots\ldots$

6. (i) show that

$$\log_{10}2+16\log_{10}\frac{16}{15}+12\log_{10}\frac{25}{24}+7\log_{10}\frac{81}{80}=1.$$

(ii) Given $\log 2=\cdot3010300$, $\log 11=1\cdot0413927$, show that Rs. 1000 will become Rs. 2500 at 10 per cent. per annum compound interest in about $9\cdot6$ years.

1941

1. (i) If α, β be the roots of the equation $x^2-px+q=0$, find the value of

$$\alpha^2\left(\frac{\alpha^2}{\beta}-\beta\right)+\beta^2\left(\frac{\beta^2}{\alpha}-\alpha\right).$$

(ii) If the difference of the roots of the equation $x^2-px+q=0$ be the same as that of the equation $x^2-qx+p=0$, show that $p+q+4=0$, unless $p=q$.

2. (i) If A varies as B when C is constant and A varies as C when B is constant, show that A varies as BC when both B and C vary.

(ii) If $x+y\propto z$ when y is constant and if $x+z\propto y$ when z is constant, show that when both y and z vary, then $x+y+z\propto yz$.

3. Solve the following equations:

(i) $\frac{xy}{x+y}=1$, $\frac{yz}{y+z}=2$, $\frac{zx}{z+x}=3$.

(ii) $x^y=y^2$, $y^{2y}=x^4$

4. (i) Find the number of combinations of n different things taken r at a time.

(ii) How many combinations can be formed of eight counters marked 1, 2, 3, 4, 5, 6, 7, 8 taking them 4 at a time, there being at least one odd and one even counter in each combination.

5. (i) Prove the Binomial theorem for a positive integral index.

(ii) Find the sum of the squares of the co-efficients in the expansion of $(1+x)$ when n is a positive integer.

6. (i) If $7^{3x+2}+4^{x+2}=7^{3x+1}+2^{2x+6}$, find by the help of the logarithmic tables the value of x [Logarithmic tables supplied].

(ii) Find the amount of an annuity of Rs. 100 in 20 years allowing compound interest at $4\frac{1}{2}$ per cent. Given $\log 1{\cdot}045={\cdot}0191163$, $\log 24{\cdot}117=1{\cdot}3823260$.

1942

1. (i) If p and q are the roots of the equation $3x^2+6x+2=0$, find the equation whose roots are $-\frac{p^2}{q}$ and $-\frac{q^2}{p}$.

(ii) Find the least value of $\frac{6x^2-22x+21}{5x^2-18x+17}$ for all real values of x.

2. Solve the equations :

(i) $x+\frac{1}{y}=\frac{3}{2}$, $y+\frac{1}{z}=\frac{7}{3}$, $z+\frac{1}{x}=4$.

(ii) $8.2^{xy}=4^y$, $9^x.3^{xy}=\frac{1}{27}$

3. (i) Find the number of permutations of n different things taken r at a time.

(ii) Find the number of numbers less than 1000 and divisible by 5 which can be formed with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, each digit not occurring more than once in each number.

4. (i) Find the co-efficient of x^{16} in the expansion of $(2x^2-x)^{10}$.

(ii) If $c_0, c_1, c_2, \cdots\cdots c_n$ denote the co-efficients in the expansion of $(1+x)^n$, prove that

$$c_1-2c_2+3c_3-\cdots\cdots+(-1)^{n-1}\,n\,c_n=0.$$

5. (i) If a, b, c, d are any four positive numbers, prove that $\log_b a \times \log_c b \times \log_d c = \log_d a$.

(ii) Find by the help of logarithmic tables the values of x and y correct to two places of decimals if

$$2^x=3^y \text{ and } 2^{y+1}=3^{x-1}.$$

6. (i) Show that

$$\frac{1}{2!}+\frac{1+2}{3!}+\frac{1+2+3}{4!}+\text{etc. to infinity}=\frac{e}{2}.$$

(ii) Find the present value of an annuity of Rs. 10000, to continue for 25 years, allowing compound interest at 5 per cent. per annum. [log 105 = 2·0211893, log 2·95362 = ·4702675]

1943

1. (i) If α and β are the roots of the equation $ax^2+bx+c=0$, find the value of $\dfrac{1}{(a\alpha+b)^3}+\dfrac{1}{(a\beta+b)^3}$.

(ii) If one root of the equation $x^2+px+q=0$ is the square of the other, show that $p^3-q\,(3p-1)+q^2=0$.

2. Solve the equations:

(i) $x(1+y)=3, \quad y(1+z)=8, \quad z(1+x)=6.$

(ii) $y^x=4, \quad y^2=2^x.$

3. (i) Find the number of combinations of n different things taken r at a time.

(ii) In a group of 15 boys there are 7 boy-scouts. In how many ways can 12 boys be selected so as to include at at least 6 boy-scouts.

4. (i) Prove the Binomial Theorem when the index is a positive integer.

(ii) Find the middle term in the expansion of $\left(x-\dfrac{1}{x}\right)^{2n}$.

5. (i) Prove the Exponential Theorem.

(ii) By the help of logarithmic tables, find the value of x, correct to 3 places of decimals, if $5^{5-3x}+4^{\frac{1}{2}x+3}=5^{7x-3x}-2^{x+}$

6. (i) Sum to infinity $1+\frac{1}{3.2^2}+\frac{1}{5.2^4}+\frac{1}{7.2^6}+\ldots$

(ii) Find the discount of Rs. 1331 due 3 years hence at 10 per cent. per annum compound interest.

1944

1. (*a*) Form the quadratic whose roots are the reciprocals of the roots of $ax^2+bx+c=0$.

(*b*) If the roots of $x^2+2px+q=0$ and $x^2+2qx+p=0$ differ by a constant, show that $p+q+1=0$.

2. (*a*) Solve : $y+z=\frac{1}{x}$, $z+x=\frac{1}{y}$, $x+y=\frac{1}{z}$.

(*b*) If x be real, prove that $\frac{2x^2-2x+4}{x^2-4x+3}$ cannot lie between 1 and -7.

3. (*a*) If $a^x=b^y=c^z$ and $b^2=ac$, prove that $\frac{1}{x}+\frac{1}{z}=\frac{2}{y}$.

(*b*) A sum of money invested at compound interest amounts to Rs. 8820 at the end of the second year and to Rs. 9261 at the end of the third year; find the sum originally invested.

4. (*a*) Find the number of permutations of n things taken all together, when things are not all different.

(*b*) Show that the letters of the word *Calcutta* can be arranged in twice as many ways as the letters of the word *America*.

5. (*a*) Express $(1+x)^n$ in the form of a series, when n is a positive integer, and calculate the sum of the co-efficients when $n=6$.

(*b*) Prove that

$$x^{\log y-\log z}\times y^{\log z-\log x}\times z^{\log x-\log y}=1.$$

6. (*a*) Expand $\log_e(1+x)$ in ascending powers of x when x is numerically less than 1.

(*b*) Sum to infinity the series $\frac{1}{2}x^2+\frac{2}{3}x^3+\frac{3}{4}x^4+\ldots\ldots$

1945

1. (a) If p and q are the roots of the equation $x^2 + 7x + 12 = 0$, find the equation whose roots are $(p+q)^2$ and $(p-q)^2$.

(b) If the roots of the equation $ax^2 + bx + c = 0$ bear to one another the ratis $3 : 4$, prove that $12b^2 = 49ac$.

2. Solve:

(a) $xy + x + y = 23$, $zx + x + z = 41$, $yz + y + z = 27$.

(b) $x^y = y^2$, $y^{2y} =$

3. (a) Find the number of combinations of n different things taken r at a time.

(b) From a company of 15 men how many selections of 9 men can be made so as to exclude three particular men?

4. (a) Prove the Binomial Theorem for a positive integral exponent.

(b) If $c_0, c_1, c_2, \ldots\ldots c_n$ be the co-efficients in the expansion of $(1+x)^n$, find the value of

$$\frac{c_0}{1} + \frac{c_1}{2} + \frac{c_2}{3} + \frac{c_3}{4} + \cdots\cdots \qquad n+1$$

5. (a) Show that

$$\frac{1}{2!} + \frac{1+2}{3!} + \frac{1+2+3}{4!} + \cdots \text{ to inf.} \quad \frac{e}{2}$$

(b) Prove that

$$\frac{}{n+1} + \frac{}{2(n+1)^2} + \frac{}{3(n+1)^3}, \qquad \frac{1}{n} \cdot \frac{1}{2n^2} + \frac{1}{3n^3} - \cdot$$

6. (a) Find (with the help of logarithmic tables) to two places of decimals the value of x from the equation

$$6^3 \qquad 4^{x+5} =$$

(b) The present value of Rs. 672 due in a certain time is Rs. 126; if compound interest at $4\frac{1}{2}$ per cent. be allowed, find the time.

PATNA UNIVERSITY

INTERMEDIATE EXAMINATION PAPERS

1944

1. (*a*) Prove that

$$\left\{\frac{-1+\sqrt{-3}}{2}\right\}^n+\left\{\frac{-1-\sqrt{-3}}{2}\right\}^n=2,$$

if n be a multiple of 3, and $=-1$, if n be any other integer.

(*b*) If $a(y+z-x)=b(z+x-y)=c(x+y-z)$, prove that

$$\frac{x}{a(b+c)}=\frac{y}{b(c+a)}=\frac{z}{c(a+b)}.$$

2. (*a*) Prove that the Arithmetic, Geometric and Harmonic means between any two unequal positive quantities are themselves in G. P. and are in descending order of magnitude.

(*b*) Solve : $\frac{1}{x-3}+\frac{1}{y-2}=5,\ 2x+3y=14.$

3. (*a*) If α, β be the roots of the equation $x^2-px+q=0$, form the equation whose roots are $\frac{q}{p-\alpha}$ and $\frac{q}{p-\beta}$

(*b*) If x be real, show that $\frac{x^2-3x-2}{2x-3}$ can have any real value.

4. (*a*) Find the number of permutations of n things taken all together when the things are not all different.

(*b*) In how many ways can the letters of the word *civilization* be re-arranged ?

5. (*a*) In the expansion of $(1+x)^n$, prove that the sum of the co-efficients of the odd terms is equal to the sum of the co-efficients of the even terms.

(*b*) If $(1+x)^n=C_0+C_1x+C_2x^2+\ldots+C_nx^n$, find the value of $C_0-\frac{C_1}{2}+\frac{C_2}{3}-\frac{C_3}{4}+\ldots\ldots+(-1)^n\frac{C_n}{n+1}$.

1945

1. (a) Prove that the two imaginary cube roots of unity are each the square of the other.

(b) If $y+z$ varies as x and $z+x$ varies as y, prove that

$$x^2+y^2+z^2 \text{ varies as } (x^3+y^3+z^3)^{\frac{2}{3}}.$$

2. (a) Insert n Harmonic Means between a and b.

(b) Sum the series $2+6+12+20+\cdots\cdots$to n terms.

3. Obtain the relations between the roots and the co-efficients of the equation $x^2-2px+q=0$. If the equation $x^2-2px+q=0$ has two equal roots, the equation $(1+y)x^2-2(p+y)x+(q+y)=0$ will have its roots real and distinct only when y is negative and p is not unity.

4. (a) Find the number of permutations of n things taken all together when the things are not all different.

(b) How many numbers between 400 and 1000 can be made with the digits 2, 3, 4, 5, 6, 0 ?

5. If $(1+x)^n=a_0+a_1x+a_2x^2+a_3x^3+\cdots\cdots\cdots+a_nx^n$, prove that

(i) $a_k=a_{n-k}$

(ii) $a_0+a_2+a_4+\cdots\cdots=a_1+a_3+a_5+\cdots\cdots$

If three successive co-efficients in the same expansion are 165, 330, and 462, find n.

ANSWERS

TO INTERMEDIATE EXAMINATION PAPERS

(Calcutta University)

1928

1. $x^2 - px + 9q - 2p^2 = 0$; $2(q+q') - pp'$.

Or, $x = \dfrac{b \pm \sqrt{a^2+ab+b^2}}{a+b}$, $y = \dfrac{a \mp \sqrt{a^2+ab+b^2}}{a+b}$;

$-\dfrac{5a^2+2ab+5b^2}{(a+b)^2}$. 2. *Or*, $\dfrac{n(n-1)}{2} - \dfrac{p(p-1)}{2} + 1$.

1929

1. (i) $1 - p - q + p^2 - pq + q^2$; (ii) $p^2 - 3q$.

Or, (i) $x = \frac{1}{5}(1 \pm 2\sqrt{5a-1})$, $y = \frac{1}{5}(2 \mp \sqrt{5a-1})$ (ii) $\frac{1}{8}$.

2. 12.

1930

1. (ii) $x = b \pm \sqrt{(a+b)^2 - c^2}$, $y = a \pm \sqrt{(a+b)^2 - c^2}$

Or, (i) $b^2 < 4ac$.

2. $n-1$. *Or*, (ii) n.

1931

1. $x = a$, $y = b$, *or* $x = y = \frac{1}{2}(a+b)$. *Or*, $\sqrt[3]{r^3 + r'^3}$.

2. $\dfrac{\lfloor n}{\lfloor p \lfloor q}$; 840.

3. $\dfrac{\lfloor 2n}{\lfloor n \lfloor n}$; $x = \dfrac{1}{2}y - \dfrac{1.3}{2.4}y^2 + \dfrac{1.3.5}{2.4.6}y^3 - \ldots\ldots\ldots$

1932

1. $\dfrac{1}{\alpha^2}$, $\dfrac{1}{\beta^2}$, if α, β are the roots of $x^2 + px + q = 0$.

$x = \dfrac{a}{a-b}$, $y = \dfrac{b}{a+b}$, or, $x = y = \dfrac{a+b}{2a}$. *Or*, 8 : 21.

2. 200.

$Or, \left(\frac{a}{b}\right)^{2n+1} + (2n+1) \left(\frac{a}{b}\right)^{2n-1} + {}^{2n+1}C_2 \left(\frac{a}{b}\right)^{2n-3} + \cdots + \left(\frac{b}{a}\right)^{2n+1}.$

General term $= {}^{2n+1}C_r \left(\frac{a}{b}\right)^{2n-2r+1}.$

Middle terms $= {}^{2n+1}C_n \frac{a}{b}$ and ${}^{2n+1}C_{n+1} \frac{b}{a}.$

1933

1. b^2 not $<4ac$; No ; $a^2x^2-(b^2-2ca)x+c^2=0.$
2. $\frac{1}{4}n^2(n+1)^2$
3. ${}^nC_r$; ${}^{n-2}C_r + {}^{n-2}C_{r-1} + {}^{n-2}C_{r-2}$; 868224.

1934

1. (ii) $-\frac{11}{8}$ or $\frac{7}{4}$. 2. $5\frac{2}{3}$ days.

3. $\frac{\lfloor n}{\lfloor p \lfloor q \lfloor r}$; $\frac{\lfloor 39}{\lfloor 5 (\lfloor 4)^2 (\lfloor 6)^3}$.

4. (ii) The 9th term $=495$. 6. (ii) 58·71 years nearly.

1935

1. (i) $\frac{b^2-2ac}{a^2}$. 2. (ii) $x=y=0$, or, $x=4$, $y=2$.

3. (ii) 25. 5. (ii) £1979. 2s. 6d. nearly.

1. (i) 2 or $\frac{1}{2}$; $-\frac{2}{3}$. (ii) $cax^2-(b^2-2ca)x+ca=0$.
2. (ii) $x=6$, $y=3$, or, $x=3$, $y=6$.
3. (i) ${}^nP_r$. (ii) 2016 ; 3024. 4. (ii) The 4th term $=\frac{1792}{9}$.
5. (i) ·8187 nearly. 6. (ii) Rs. 1335. 9 as. 2·8p nearly.

1937

2. (i) $x=\pm 4$. (ii) $x=\frac{1}{3}$, $y=\frac{1}{2}$; or, $x=\frac{5}{2}$, $y=-\frac{5}{3}$.
3. (i) ${}^nC_r$. (ii) 246. 4. (ii) 121.
5. (ii) £785. 2s. 2d. nearly. 6. (i) $1-\frac{x^2}{\lfloor 2}+\frac{x^4}{\lfloor 4}-\cdots\cdots$

1938

1. (i) $x^2-2(p^2-2q)x+p^2(p^2-4q)=0$.
2. (i) $x=2$ or 8, $y=8$ or 2. (ii) $x=\pm 1$, $y=\pm 2$, $z=\pm 3$.

3. (ii) 344. 4. (i) $\dfrac{1.3.5 \ldots (2r-1)}{\lfloor r}$.

5. (i) $\left(1+1+\dfrac{1}{\lfloor 2}+\dfrac{1}{\lfloor 3}+\cdots\right)+\left(1+\dfrac{2}{\lfloor 2}+\dfrac{3}{\lfloor 3}+\cdots\right)x$

$+\dfrac{1}{\lfloor 2}\left(1+\dfrac{2^2}{\lfloor 2}+\dfrac{3^2}{\lfloor 3}+\cdots\right)x^2+\dfrac{1}{\lfloor 3}\left(1+\dfrac{2^3}{\lfloor 2}+\dfrac{3^3}{\lfloor 3}+\cdots\right)x^3$

$+\dfrac{1}{\lfloor 4}\left(1+\dfrac{2^4}{\lfloor 2}+\dfrac{3^4}{\lfloor 3}+\cdots\right)x^4+\ldots\ldots$

(ii) $e+e^{-1}-2$. 6. (i) $x=1.77$ nearly.

(ii) 14·2 years nearly.

1939

1. (i) $cax^2+b(c+a)x+(c+a)^2=0$.
2. (i) $x=3$ or 6, $y=6$ or 3;
   (ii) $x=4$ or $\frac{60}{7}$, $y=6$ or $\frac{66}{7}$, $z=2$ or -6.
3. (ii) 816000. 4. (ii) $4n$. 5. (ii) $5e$.
6. (ii) Rs. 31200.

1940

2. (i) $x=\frac{1}{5}$ or $\frac{4}{5}$, $y=5$ or 20; (ii) $x=2$ or 8, $y=4$ or 4, $z=8$ or 2. 3. (ii) 576. 4. (i) ${}^nC_{\frac{n}{2}}x^{\frac{n}{2}}$ (n even), ${}^nC_{\frac{n-1}{2}}x^{\frac{n-1}{2}}$ and ${}^nC_{\frac{n+1}{2}}x^{\frac{n+1}{2}}$ (n odd). 5. (ii) $1-\log_e 2$.

1941

1. (i) $\dfrac{p}{q}(p^4-5p^2q+4q^2)$. 3. (i) $x=\dfrac{12}{5}$, $y=\dfrac{12}{7}$, $z=-12$; (ii) $x=2$, $y=2$. 4. (ii) 68. 5. (ii) $\dfrac{\lfloor 2n}{\lfloor n \lfloor n}$.
6. (i) $x=·03$ nearly. (ii) Rs. 5359. 5 as. 4 p.

1942

1. (i) $3x^2-18x+2=0$; (ii) 1. 2. (i) $x=1$ or $\frac{3}{10}$, $y=2$ or $\frac{5}{6}$, $z=3$ or $\frac{2}{3}$; (ii) $x=-1$ or 3, $y=1$ or -3. 3. (ii) 154.
4. (i) 13440. 5. (ii) $x=2·71$ nearly, $y=1·71$ nearly.
6. (ii) Rs. 140927. 9as. 2·7p. nearly.

1943

1. (i) $\frac{b^3-3abc}{a^3c^3}$.

2. (i) $x=1, y=2, z=3$; or $x=-\frac{7}{3}, y=-\frac{16}{7}, z=-\frac{9}{2}$.
   (ii) $x=2, y=\pm 2$, or $x=-2$; $y=\pm\frac{1}{2}$.

3. (ii) 252. 4. (ii) $(-1)^n \frac{\lfloor 2n}{\lfloor n \lfloor n}$, 5. (ii) 1·206.

6. (i) $\log_e 3$: (ii) 331 rupees.

1944

1. (a) $cx^2+bx+a=0$. 2. (a) $x=y=z=\frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}$.

3. (b) Rs. 8000. 5. (a) 64. 6. (b) $\frac{x}{1-x}+\log(1-x)$.

1945

1. (a) $x^2-50x+49=0$. 2. (a) $x=5$ or -7, $y=3$ or -5, $z=6$ or -8. (b) $x=2, y=2$.

3. (b) 220. 4. (b) $\frac{2^{n+1}-1}{n+1}$. 6. (a) $x=1·77$ nearly ;
(b) 41 years nearly.

Patna University

1944

2. (b) $x=3\frac{1}{3}, 3\frac{2}{5}$; $y=2\frac{1}{3}, 2\frac{2}{5}$.
3. (a) $x^2-px+q=0$.
4. (b) 19958399. 5 (b) $\frac{1}{n+1}$.

1945

2. (a) see Art. 188 (c). (b) $\frac{1}{3}n(n+1)(n+2)$.
3. $\alpha+\beta=2p$, $\alpha\beta=q$.
4. (b) 60. 5. $n=11$.

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